MATHEMATICS: COMPLETE REVISION & PRACTICE SSIP: NSC EXAM KIT 2020

CONTENTS

Mark distribution for Mathematics NCS end - of - year papers: Grade 12	
PAPER 1: GRADE 12 :bookwork: maximum 6 marks	
Description	**Grade 12**
Algebra and equations(and inequalities)	25 ± 3
Patterns and sequence	25 ± 3
Finance and Growth	15 ± 3
Functions and graphs	35 ± 3
Differential calculus	35 ± 3
Probability	15 ± 3
TOTAL	**150**

SECTION1
ALGEBRA, EQUATIONS & INEQUALITIES

In this topic, a candidate is expected to apply all the necessary algebraic techniques/concepts like simplifying of expressions and equations, using the **BODMAS** rule, exponential laws, among all other algebraic manipulation techniques.

The sum and difference of two cubes

The sum of two cubes

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

The difference of two cubes

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

The difference of squares

$$x^2 - y^2 = (x-y)(x+y)$$

EXPONENTIAL LAWS

	Law	Example
1.	$a^m . a^n = a^{m+n}$	$3^3 . 3^4 = 3^7$
2.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^8}{2^4} = 2^{8-4} = 2^4 = 16$
3.	$(a^m)^n = a^{m\times n} = (a^n)^m$	$(3^3)^4 = 3^{3\times 4} = 3^{12} = (3^4)^3$
4.	$(ab)^m = a^m b^m$	$(3x^4y^3)^2 = 3^{1\times 2} . x^{4\times 2} . y^{3\times 2} = 9x^8y^6$
5.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{4}{x^2}\right)^3 = \frac{4^{1\times 3}}{x^{2\times 3}} = \frac{64}{x^6}$

Definition	Example
1. $a^0 = 1$	$3^0 = 1$
2. $1^a = 1$, with $a \in R$	$1^{2008} = 1$
3. $x^{-n} = \dfrac{1}{x^n}$	$x^{-4} = \dfrac{1}{x^4}$
4. $ax^{-n} = \dfrac{a}{x^n}$	$3x^{-4} = \dfrac{3}{x^4}$
5. $(ax)^{-n} = \dfrac{1}{(ax)^n}$	$(3x)^{-4} = \dfrac{1}{(3x)^4}$
6. $\dfrac{1}{x^{-n}} = x^n$	$\dfrac{1}{x^{-4}} = x^4$
7. $\dfrac{a}{x^{-n}} = ax^n$	$\dfrac{3}{x^{-4}} = 3x^4$
8. $\dfrac{1}{ax^{-n}} = \dfrac{x^n}{a}$	$\dfrac{1}{3x^{-4}} = \dfrac{x^4}{3}$
9. $\dfrac{1}{(ax)^{-n}} = (ax)^n$	$\dfrac{1}{(3x)^{-4}} = (3x)^4$
10. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$
11. $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$	$\dfrac{x^{-3}}{y^{-4}} = \dfrac{y^4}{x^3}$

SURDS

When a root is irrational it is referred to as a surd. In other words a surd is the root of a number that cannot be determined exactly. Thus $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{20}$ are surds but $\sqrt{4}$, $\sqrt{36}$ and $\sqrt[3]{27}$ are not.

There are two surd laws for $a > 0$, $b > 0$, $n \geq 2$, $n \in$natural numbers.

Law 1 (multiplication law): $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$ or $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

Law 2 (division law) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ or $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

SOLVING BY FACTORIZATION

Example 1

$3x^2 + 5x = 2$	
$3x^2 + 5x - 2 = 0$	✓ standard form
$(3x-1)(x+2) = 0$	✓ factors
$x = \frac{1}{3} \ldots OR \ldots x = -2$	✓ both x values

USING THE QUADRATIC FORMULA

Example 1

$x^2 + x - 13 = 0$	
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	
$= \frac{-1 \pm \sqrt{1 - 4(1)(-13)}}{2}$	✓ subs into formula
$= \frac{-1 \pm \sqrt{53}}{2}$	✓ $\sqrt{53}$
$x = 3{,}14$ or $x = -4{,}14$	✓answer ✓answer

SIMPLIFYING EXPRESSIONS

Example 1

$\frac{2^{2015} + 2^{2013}}{4^{1006}}$	
$= \frac{2^{2013}(2^2 + 1)}{2^{2012}}$	✓factors ✓2^{2012}
$= 2(5)$	
$= 10$	✓answer

Example 2

Calculate a and b if $\sqrt{\dfrac{7^{2014}-7^{2012}}{12}}=a\left(7^{b}\right)$ and a is not a multiple of 7.

$\sqrt{\dfrac{7^{2014}-7^{2012}}{12}}$	
$=\sqrt{\dfrac{7^{2012}\left(7^{2}-1\right)}{12}}$	✓ $\dfrac{7^{2012}\left(7^{2}-1\right)}{12}$
$=\sqrt{\dfrac{7^{2012}.48}{12}}$	
$=\sqrt{7^{2012}.4}$	✓ $\sqrt{7^{2012}.4}$
$=2.7^{1006}$	✓ 2.7^{1006} ✓
$a=2;\ b=1006$	**OR/*OF*** ✓ $a=2$ ✓ $b=1006$

Example 3

Simplify, without the use of a calculator:

$$\sqrt{3}.\sqrt{48}-\frac{4^{x+1}}{2^{2x}}$$

solution

$\sqrt{3}.\sqrt{48}-\dfrac{4^{x+1}}{2^{2x}}$	
$=\sqrt{3}.4\sqrt{3}-\dfrac{2^{2x+2}}{2^{2x}}$	✓ 2^{2x+2}
$=12-4$	✓ 4
$=8$	✓ answer
OR	
$\sqrt{3}.\sqrt{48}-\dfrac{4^{x+1}}{2^{2x}}$	
$=\sqrt{144}-\dfrac{2^{2x+2}}{2^{2x}}$	✓ 2^{2x+2}
$=12-4$	✓ 4
$=8$	✓ answer

EXPONENTIAL EQUATIONS

Example 1

$3^x + 3^{-x+1}.5 = 8$	
$3^x + \dfrac{3.5}{3^x} = 8$	
$3^{2x} - 8.3^x + 15 = 0$	✓standard form
$(3^x - 5)(3^x - 3) = 0$	✓✓factors
$3^x = 5 \quad OR \quad 3^x = 3$	
$x = \log_3 5 \qquad x = 1$	✓ $x = 1$
$x = 1{,}46$	✓ $x = \log_3 5$ OR $x = 1{,}46$

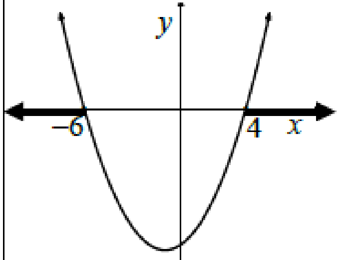
EQUATIONS WITH SURDS

Example 1

$\sqrt{x+7} - 1 = x$	
$\sqrt{x+7} = x + 1$	
$x + 7 = (x+1)^2$	✓ $x + 7 = (x+1)^2$
$x + 7 = x^2 + 2x + 1$	
$x^2 + x - 6 = 0$	✓standard form
$(x+3)(x-2) = 0$	✓factors
$x = -3 \quad OR \quad x = 2$	
N/A.	✓N/A
$\therefore x = 2$	✓ $x = 2$

INEQUALITIES

Example 1

$x^2 + 2x - 24 \geq 0$	
$(x+6)(x-4) \geq 0$	✓ factors
 **OR** 	
$x \leq -6$ or $x \geq 4$	✓✓ $x \leq -6$ or $x \geq 4$

SIMULTANEOUS EQUATIONS

Example 1

$$x - 2y = 3 \quad \ldots\ldots 1$$
$$4x^2 - 5xy = 3 - 6y \quad \ldots\ldots 2$$

$x - 2y = 3$
$x = 2y + 3$ ✓ $x = 2y + 3$

$4x^2 - 5xy = 3 - 6y$
$4(2y + 3)^2 - 5y(2y + 3) = 3 - 6y$ ✓ substitution/s
$4(4y^2 + 12y + 9) - 10y^2 - 15y = 3 - 6y$
$16y^2 + 48y + 36 - 10y^2 - 15y = 3 - 6y$
$6y^2 + 39y + 33 = 0$ ✓ standard form
$2y^2 + 13y + 11 = 0$
$(2y + 11)(y + 1) = 0$ ✓ factors
$y = \frac{-11}{2}$ or/of $y = -1$ ✓ y-values
$x = -8$ or/of $x = 1$ ✓ x-values

Example 2

Solve for x and y if: $3^{x-10} = 3^{3x}$ and $y^2 + x = 20$.

$3^{x-10} = 3^{3x}$
$x - 10 = 3x$ ✓ equating of exponents
$2x = -10$
$x = -5$ ✓ x-value

$y^2 + x = 20$
$y^2 - 5 = 20$ ✓ $20 = y^2 - 5$
$y^2 = 25$
$y = -5$ or $y = 5$ ✓✓ y-values

THE NATURE OF ROOTS FOR A QUADRATIC EQUATION

Consider the quadratic equation: $ax^2 + bx + c = 0$,
The solution to this equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $\Delta = b^2 - 4ac$, we can determine the nature of roots using the following conditions

$b^2 - 4ac$	Roots
$\Delta < 0$	Non-real
$\Delta \geq 0$	Real
$\Delta > 0$ and Δ = perfect square	Real, rational and unequal
$\Delta > 0$ and $\Delta \neq$ a perfect square	Real, irrational and unequal
$\Delta = 0$	Real, rational and equal

Example 1

Given: $f(x) = 3(x-1)^2 + 5$ and $g(x) = 3$

Determine the value(s) of k for which $f(x) = g(x) + k$ has TWO unequal real roots.

$3x^2 - 6x + 3 + 5 = 3 + k$

$3x^2 - 6x + 5 - k = 0$

$$\Delta = (-6)^2 - 4(3)(5-k)$$
$$= 36 - 60 + 12k$$
$$= 12k - 24$$

For real unequal roots

$$12k - 24 > 0$$
$$12k > 24$$
$$k > 2$$

Example 2

Given: $f(x) = x^2 - 5x + c$

Determine the value of c if it is given that the solutions of $f(x) = 0$ are $\dfrac{5 \pm \sqrt{41}}{2}$.

$f(x) = x^2 - 5x + c$	
$x = \dfrac{5 \pm \sqrt{25 - 4(1)(c)}}{2}$	✓substitution into correct formula
$25 - 4c = 41$	✓ $25 - 4c = 41$
$-4c = 16$	
$c = -4$	✓ $c = -4$

Misconceptions and common errors

1.	Incorrect rounding off when using the quadratic formula. e.g $$3x^2+5x-7=0$$ $$x=\frac{-(5)\pm\sqrt{(5)^2-4(3)(-7)}}{2(3)}$$ $x=-2{,}57$ ✓ OR $x=0{,}90$ wrong the acceptable value is $x=0{,}91$
2.	Simultaneous equations. Learners forget to substitute for the x −values in order to get the corresponding y −values.
3.	Equations with surds and squaring of both sides of the equation to eliminate the square root. This operation is usually done incorrectly Eg $\sqrt{x+7}=x+2$ $$x+7=(x+2)^2 ✓$$ $$x+7=x^2+4 \text{ wrong}$$
4.	Learners fail to standardize quadratic equations when solving for x. e.g $$3x^2-7=-5x$$ In this case a learner considers $a=3$ ✓ ; $b=-7$ wrong and $c=-5$ wrong The correct equation in standard form should be; $$3x^2+5x-7=0$$ Whereby $a=3; b=5 \text{ and } c=-7$
5.	Mishandling signs especially when substituting the quadratic formula.
6.	Solving inequalities. When dividing by a negative value, learners tend to forget to change the inequality sign. e.g $$-3x>10$$ $$x>-\frac{10}{3} \text{ wrong}$$ Correction $$-3x>10$$ $$x<-\frac{10}{3} ✓$$

Tips:

- When solving a quadratic equation, make sure that it is in a standard form(i.e the equation equated to 0)
- When solving simulatenous equations, always calculate both x *values* and y *values*. if your first values are for x, then substitute them back into the original equation to get the corresponding y *values*.
- When solving equations with surds or fractions, always remember to check your results to determine which result satisfies the equations.
- Always round off your answers to the correct number of decimal places as demanded by the question. Eg 1 d.p or 2 d.p

- There are three methods for solving a quadratic equation. Factorization, quadratic formula and completing squares. It's important that you understand which method is approciate to the question.

EXAMINATION QUESTIONS ON ALGEBRA

Feb- March 2018

QUESTION 1

1.1 Solve for x:

1.1.1 $x^2 - 6x - 16 = 0$ (3)

1.1.2 $2x^2 + 7x - 1 = 0$ (correc (4)

1.2 List all the integers that are solutions to $x^2 - 25 < 0$. (4)

1.3 Solve for x and y:

$-2y + x = -1$ and $x^2 - 7 - y^2 = -y$ (6)

1.4 Evaluate: $\dfrac{3^{2018} + 3^{2016}}{3^{2017}}$ (2)

1.5 Given: $t(x) = \dfrac{\sqrt{3x-5}}{x-3}$

1.5.1 For which values of x will $\dfrac{\sqrt{3x-5}}{x-3}$ be real? (3)

1.5.2 Solve for x if $t(x) = 1$. (4)

[26]

Feb /March 2017

QUESTION 1

1.1 Solve for x:

1.1.1 $(x-3)(x+1)=0$ (2)

1.1.2 $\sqrt{x^3}=512$ (3)

1.1.3 $x(x-4)<0$ (2)

1.2 Given: $f(x)=x^2-5x+2$

1.2.1 Solve for x if $f(x)=0$ (3)

1.2.2 For which values of c will $f(x)=c$ have no real roots? (4)

1.3 Solve for x and y:

$$x=2y+2$$
$$x^2-2xy+3y^2=4$$ (6)

1.4 Calculate the maximum value of S if $S=\dfrac{6}{x^2+2}$. (2)

[22]

Nov 2018

QUESTION 1

1.1 Solve for x:

1.1.1 $x^2-4x+3=0$ (3)

1.1.2 $5x^2-5x+1=0$ (correct to TWO decimal places) (3)

1.1.3 $x^2-3x-10>0$ (3)

1.1.4 $3\sqrt{x}=x-4$ (4)

1.2 Solve simultaneously for x and y:

$3x-y=2$ and $2y+9x^2=-1$ (6)

1.3 If $3^{9x}=64$ and $5^{\sqrt{p}}=64$, calculate, WITHOUT the use of a calculator, the value of: $\dfrac{\left[3^{x-1}\right]^3}{\sqrt{5^{\sqrt{p}}}}$ (4)

[23]

Nov 2019

1.1 Solve for x:

1.1.1 $x^2 + 5x - 6 = 0$ (3)

1.1.2 $4x^2 + 3x - 5 = 0$ (correct to TWO decimal places) (3)

1.1.3 $4x^2 - 1 < 0$ (3)

1.1.4 $\left(\sqrt{\sqrt{32}+x}\right)\left(\sqrt{\sqrt{32}-x}\right) = x$ (4)

1.2 Solve simultaneously for x and y:
$y + x = 12$ and $xy = 14 - 3x$ (5)

1.3 Consider the product $1 \times 2 \times 3 \times 4 \times \ldots \times 30$.
Determine the largest value of k such that 3^k is a factor of this product. (4)

[22]

SECTION 2
SEQUENCES AND SERIES

SEQUENCE;

This is a particular order in which related terms follow each other.

$$T_1\ ; T_2;\ T_3;\ T_4; \ldots$$

SERIES;

This is the addition of terms of the sequence.

$$T_1 + T_2 + T_3 + T_4 + \cdots$$

In Grade 10 and 11 you learnt about **linear and quadratic** number patterns. Linear number patterns have a **constant difference** between consecutive terms while quadratic number patterns have a **constant second difference**.

QUADRATIC NUMBER PATTERNS

$T_n = an^2 + bn + c$.

The terms of the number pattern would then be:

$T_1 = a(1)^2 + b(1) + c = a + b + c$

$T_2 = a(2)^2 + b(2) + c = 4a + 2b + c$

$T_3 = a(3)^2 + b(3) + c = 9a + 3b + c$

$T_4 = a(4)^2 + b(4) + c = 16a + 4b + c$

$a+b+c \qquad 4a+2b+c \qquad 9a+3b+c \qquad 16a+4b+c$

$3a+b \qquad 5a+b \qquad 7a+b$ → **First difference**

$2a \qquad 2a$ → **Second difference**

You will notice that the constant second difference is given by the expression $2a$. The first term in the first difference row is given by $3a + b$ and the first term is given by $a + b + c$.

Example 1

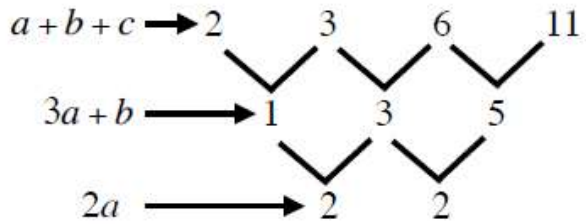
Consider the following number pattern: $2;3;6;11;\ldots$

(a) Determine the nth term (general term) and hence the value of the 42nd term.

(b) Determine which term will equal 1091.

Solutions

(a)



$2a = 2$ $\quad$ $3a+b=1$ $\quad$ $a+b+c=2$

$\therefore a=1$ $\quad$ $\therefore 3(1)+b=1$ $\quad$ $\therefore (1)+(-2)=2$

$\therefore b=-2$ $\quad$ $\therefore c=3$

$\therefore T_n = n^2-2n+3$

$\therefore T_{42} = (42)^2-2(42)+3 = 1683$

(b) $T_n = 1091$

$\therefore n^2-2n+3=1091$

$\therefore n^2-2n-1088=0$

$\therefore (n-34)(n+32)=0$

$\therefore n=34$ or $n=-32$

But $n \neq -32$

$\therefore n=34$

The 34th term will equal 1091

Example 2

Consider the quadratic sequence: $-6;\ 2-x;\ 4;\ 2x+5$

1. Solve for x.
2. Determine the n^{th} term in the sequence.
3. Which term has a value of 400?

Solutions

1 $\quad -6 \qquad 2 - x \qquad 4 \qquad 2x + 5$

$8 - x \qquad 2 + x \qquad 2x + 1$

$2x - 6 \qquad x - 1$

$2x - 6 = x - 1 \Rightarrow x = 5$

The sequence is $-6; -3; 4; 15$

2 $\quad T_n = an^2 + bn + c$

T_1		T_2		T_3		T_4
$\boxed{-6}$		-3		4		15
	$\boxed{3}$		7		11	
		$\boxed{4}$		4		

$2a = 4 \Rightarrow a = 2$

$3a + b = 3 \Rightarrow b = 3 - 6 = -3$

$a + b + c = -6$

$2 - 3 + c = -6$ and $c = -5$

$T_n = 2n^2 - 3n - 5$

3 $\quad 2n^2 - 3n - 5 = 400 \Rightarrow 2n^2 - 3n - 405 = 0$

$(2n + 27)(n - 15) = 0$ | Use the quadratic formula if necessary.

$n \neq -\frac{27}{2}$, so $n = 15$ only | $n \in \mathbb{N}$

The 15^{th} term has a value of 400.

ARITHMETIC/LINEAR SEQUENCES

Linear patterns are also called **arithmetic sequences** and have a general term of $T_n = a + (n-1)d$ where:

- a represents the first term
- d represents the constant or common difference
- n represents the position of the term
- T_n represents the nth term or general term (the value of the term in the nth position)

An arithmetic sequence is characterised by a constant difference(d).
i.e

$$d = T_2 - T_1 = T_3 - T_2$$

Example

$x\,;\,4x+5\,;\,10x-5$ are the first three terms of an arithmetic sequence. Determine the value of x and hence the sequence.

Solution

Since the sequence is arithmetic, it is clear that

$d = T_2 - T_1 = (4x+5) - (x) = 3x - 5$

$d = T_3 - T_2 = (10x-5) - (4x+5) = 10x - 5 - 4x - 5 = 6x - 10$

$\therefore 3x + 5 = 6x - 10$

$\therefore -3x = -15$

$\therefore x = 5$

$\therefore T_1 = x = 5$

$\therefore T_2 = 4x + 5 = 4(5) + 5 = 25$

$\therefore T_3 = 10x - 5 = 10(5) - 5 = 45$

$\therefore$ The sequence is $5\,;\,25\,;\,45\,;\,\ldots\ldots\ldots$

GEOMETRIC/ EXPONENTIAL SEQUENCES

The general formula is

$$T_n = ar^{n-1}$$

where:

- a represents the first term
- d represents the constant or common ratio
- n represents the position of the term
- T_n represents the nth term or general term (the value of the term in the nth position)

Geometric sequences are characterised by a constant ratio(r).

i.e

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

Example

$k+1\ ;\ k-1\ ;\ 2k-5$ are the first three terms of a geometric sequence. Calculate the value of k and hence determine the possible sequences.

Solutions

$$\frac{T_2}{T_1} = \frac{T_3}{T_2}$$

$$\therefore \frac{k-1}{k+1} = \frac{2k-5}{k-1}$$

$$\therefore (k-1)^2 = (2k-5)(k+1)$$

$$\therefore k^2 - 2k + 1 = 2k^2 - 3k - 5$$

$$\therefore 0 = k^2 - k - 6$$

$$\therefore 0 = (k-3)(k+2)$$

$$\therefore k = 3 \text{ or } k = -2$$

For $k = 3$:

$T_1 = k + 1 = 3 + 1 = 4$

$T_2 = k - 1 = 3 - 1 = 2$

$T_3 = 2k - 5 = 2(3) - 5 = 1$

The sequence is therefore:

$4\ ;\ 2\ ;\ 1\ ;\ ....$

For $k = -2$:

$T_1 = k + 1 = -2 + 1 = -1$

$T_2 = k - 1 = -2 - 1 = -3$

$T_3 = 2k - 5 = 2(-2) - 5 = -9$

The sequence is therefore:

$-1\ ;\ -3\ ;\ -9\ ;\ ....$

SERIES AND SIGMA NOTATION

The mathematical symbol $\sum$ is the capital letter S in the Greek alphabet. It is used as the symbol for summing a series. Consider the following series:

$$\sum_{k=1}^{n} \mathrm{T}_k = \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3 + \ldots + \mathrm{T}_n$$

This is read as follows:
The sum of all the terms T_k (general term) from $k = 1$ to $k = n$ where $n \in \mathrm{N}$.

Example 1

Calculate: $\sum_{k=1}^{7} (5k - 3)$

Solution

<u>Last</u> value to be substituted into the general term

General term

$$\sum_{k=1}^{7} (5k-3) = [5(1)-3]+[5(2)-3]+[5(3)-3]+[5(4)-3]+[5(5)-3]+[5(6)-3]+[5(7)-3]$$

$$= 2+7+12+17+22+27+32$$

$$= 119$$

<u>First</u> value to be substituted into the general term

Example 2

(a) $\sum_{k=1}^{8} 2k = 2(1)+2(2)+2(3)+2(4)+2(5)+2(6)+2(7)+2(8)$

$= 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$ **Number of terms:** 8

$= 72$ $(8-1+1=8)$

(b) $\sum_{k=0}^{8} 2k = 2(0)+2(1)+2(2)+2(3)+2(4)+2(5)+2(6)+2(7)+2(8)$

$= 0 + 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$ **Number of terms:** 9

$= 72$ $(8-0+1=9)$

(c) $\sum_{k=2}^{8} 2k = 2(2)+2(3)+2(4)+2(5)+2(6)+2(7)+2(8)$

$= 4 + 6 + 8 + 10 + 12 + 14 + 16$ **Number of terms:** 7

$= 70$ $(8-2+1=7)$

Therefore in general for $\sum_{k=m}^{n} T_k$:

$$\boxed{\text{Number of terms} = \text{Top} - \text{Bottom} + 1 = n - m + 1}$$

SUMMING THE TERMS OF AN ARITHMETIC SERIES

An arithmetic series is the sum of an arithmetic sequence. $5\,;7\,;9\,;11\,;\ldots$ is an arithmetic sequence and $5 + 7 + 9 + 11 + \ldots.$ is the sum of that sequence.

We will use the formulae below to calculate the sum of a finite arithmetic series:

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}\left[a + l\right] \quad (\,l \text{ is the last term of the series}) \\ l = T_n = a + (n-1)d}$$

Example 1

Find the sum of the first 20 terms of the series $-1 + 3 + 7 + \ldots.$

<u>Solution</u>

The given series is arithmetic with $a = -1$ and $d = 4$. $S_{20} = ?$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ (state the formula)

$\therefore S_{20} = \frac{20}{2}\left[2(-1) + (20-1)(4)\right]$ (substitute $n = 20$, $a = -1$ and $d = 4$)

$\therefore S_{20} = 10(-2 + 76) = 740$

Example 2

Calculate $\sum_{m=2}^{100}(7-2m)$

Solution

Expand so as to identify the type of series:

$$\sum_{m=2}^{100}(7-2m)=[7-2(2)]+[7-2(3)]+[7-2(4)]+[7-2(5)]+\ldots\ldots+[7-2(100)]$$

$$=3+1+(-1)+(-3)+\ldots.+(-193)$$

This is an arithmetic series with $a=3$ and $d=T_2-T_1=T_3-T_2=-2$ and $l=-193$
Now determine the number of terms:

$n=100-2+1=99$ (Number of terms = Top – Bottom + 1)

$$\boxed{\therefore \sum_{m=2}^{100}(7-2m) \quad \Rightarrow \quad S_{99}=?}$$ (an important observation)

$S_n=\frac{n}{2}(a+l)$ or $S_n=\frac{n}{2}[2a+(n-1)d]$

Substitute $n=99$, $a=3$ and $l=-193$

$$\therefore S_{99}=\frac{99}{2}(3+(-193))$$

$$\therefore S_{99}=-9405$$

Substitute $n=99$, $a=3$ and $d=-2$

$$\therefore S_{99}=\frac{99}{2}[2(3)+(99-1)(-2)]$$

$$\therefore S_{99}=-9405$$

SUMMING THE TERMS OF A GEOMETRIC SERIES

A geometric series is the **sum** of a geometric sequence. $3\,;6\,;12\,;24\,;\ldots$ is a geometric sequence and the sum of that sequence, $3 + 6 + 12 + 24 + \ldots.$ is a geometric series.

We will use the formulae below to calculate the sum of a finite geometric series:

$$S_n = \frac{a\left(r^n - 1\right)}{r-1},\ r \neq 1 \qquad \text{or} \qquad S_n = \frac{a\left(1 - r^n\right)}{1-r},\ r \neq 1$$

Example 1

Calculate the sum of the following finite series $0,25 + 0,5 + 1 + 2 + \ldots. + 256$.

Solution

It is necessary to first calculate the number of terms in the series before being able to determine the sum of the series.

$a = 0,25,\ r = 2$ and $T_n = 256$

$T_n = ar^{n-1}$ (state the formula)

$\therefore 256 = (0,25)(2)^{n-1}$ (substitute $T_n = 256,\ a = 0,25$ and $r = 2$)

$\therefore \dfrac{256}{0,25} = 2^{n-1}$

$\therefore 1024 = 2^{n-1}$

$\therefore 2^{10} = 2^{n-1}$

$\therefore 10 = n - 1$

$\therefore n = 11$

$\therefore$ There are 11 terms in the series.

In order to calculate the sum of the series, either one of the two formulae may be used:

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \qquad \text{or} \qquad S_n = \frac{a\left(1 - r^n\right)}{1-r}$$

$$\therefore S_{11} = \frac{0,25\left((2)^{11} - 1\right)}{(2)-1} \qquad \qquad \therefore S_{11} = \frac{0,25\left(1 - (2)^{11}\right)}{1-(2)}$$

$$\therefore S_{11} = 511,75 \qquad \qquad \therefore S_{11} = 511,75$$

Example 2

Calculate $\sum_{i=0}^{19} 3.(-2)^{i-1}$

Solution

$$\sum_{i=0}^{19} 3.(-2)^{i-1} = \left[3.(-2)^{0-1}\right]+\left[3.(-2)^{1-1}\right]+\left[3.(-2)^{2-1}\right]+\ldots\ldots+\left[3.(-2)^{19-1}\right]$$

$$= -\frac{3}{2} + 3 + (-6) + \ldots. + 786\ 432$$

This is a geometric series with $a = -\frac{3}{2}$ and $r = -2$

The number of the terms is:

$n = 19 - 0 + 1 = 20$ $\quad$ (Number of terms = Top − Bottom + 1)

$$\boxed{\sum_{i=0}^{19} 3.(-2)^{i-1} \quad \Rightarrow \quad S_{20} = ?}$$

$S_n = \dfrac{a(r^n - 1)}{r - 1}$ $\quad$ (state the formula and substitute $a = -\frac{3}{2}$, $r = -2$ and $n = 20$)

$$\therefore S_{20} = \frac{\left(-\frac{3}{2}\right)\left((-2)^{20} - 1\right)}{(-2) - 1} = 524\ 287,5$$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

Example 1

The sum of the first 12 terms of an arithmetic series is 96. The 3rd and 6th terms add up to 12. Determine the first term and the common difference.

Solution

$S_{12} = 96$ and $T_3 + T_6 = 12$

$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$\therefore \frac{12}{2}\left(2a + (12-1)d\right) = 96$

$\therefore 6\left(2a + 11d\right) = 96$

$\therefore 2a + 11d = 16 \; \ldots . \text{A}$

$T_n = a + (n-1)d$

$\therefore \left(a + (3-1)d\right) + \left(a + (6-1)d\right) = 12$

$\left(a + 2d\right) + \left(a + 5d\right) = 12$

$2a + 7d = 12 \ldots . \text{B}$

Solve simultaneously:

$$\begin{array}{ll} 2a + 11d = 16 \; \ldots . \text{A} & \\ \underline{2a + 7d = 12 \; \ldots . \text{B}} & \\ \therefore 4d = 4 & \text{A} - \text{B} \\ \therefore d = 1 & \end{array}$$

Substitute $d = 1$ into A

$\therefore 2a + 11(1) = 16$

$\therefore 2a + 11 = 16$

$\therefore 2a = 5$

$\therefore a = \frac{5}{2}$

$\therefore$ The first term is $\frac{5}{2}$ and the constant difference is 1

Example 2

Determine the first three terms of the geometric sequence of which the 7th term is 1458 and the 4th term is 54.

Solution

_ ; _ ; _ ; 54 ; _ ; _ ; 1458 ; .....

To find the first three terms you have to find the value of a and the constant ratio r to generate the sequence.

$T_4 = 54$ and $T_7 = 1458$

$T_n = ar^{n-1}$ (general term of a geometric sequence)

$T_4 = ar^{4-1}$ and $T_7 = ar^{7-1}$

$\therefore ar^{4-1} = 54$ $\quad$ $\therefore ar^{7-1} = 1458$

$\therefore ar^3 = 54 \; \ldots. \text{A}$ $\quad$ $\therefore ar^6 = 1458 \; \ldots. \text{B}$

Solving simultaneously:

$\therefore a.r^6 = 1458 \; \ldots. \text{B}$

$\therefore a.r^3 = 54 \; \ldots. \text{A}$

$\therefore r^3 = 27 \; \ldots. \text{B} \div \text{A}$

$\therefore r = \sqrt[3]{27}$

$\therefore r = 3$

$\therefore a(3)^3 = 54$ (substitute $r = 3$ into A)

$\therefore 27a = 54$

$\therefore a = 2$

$\therefore$ The first three terms are: $2;\; 2\times 3;\; 2\times 3^2 = 2\,;6\,;18$

THE SUM TO INFINITY OF A CONVERGENT GEOMETRIC SERIES

Case 1 In a **convergent** geometric series where $-1 < r < 1$, the sum to infinity exists.

Case 2 In a **divergent** geometric series where $r < -1$ or $r > 1$, the sum to infinity doesn't exist.

In a convergent geometric series in which $-1 < r < 1$, the sum to infinity is given by the formula: $S_\infty = \dfrac{a}{1-r}$

Example 1

Calculate the sum to infinity of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$.

Solution

The series is geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$

Since $-1 < \frac{1}{2} < 1$, the series converges and the sum to infinity exists.

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)} = 1 \qquad \text{(substitute } a = \tfrac{1}{2} \text{ and } r = \tfrac{1}{2}\text{)}$$

Example 2

Calculate: $\displaystyle\sum_{n=1}^{\infty} 2.10^{1-n}$ (if it exists)

Solution

$$\sum_{n=1}^{\infty} 2.10^{1-n} = \left[2.10^{1-1}\right] + \left[2.10^{1-2}\right] + \left[2.10^{1-3}\right] + \ldots.$$

$$= 2 + 0{,}2 + 0{,}02 + \ldots.$$

This is an infinite geometric series with $a = 2$ and $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = 0{,}1$

Since $-1 < 0{,}1 < 1$, the series converges.

$$S_\infty = \frac{a}{1-r} = \frac{2}{1-0{,}1} = \frac{20}{9} = 2{,}\dot{2}$$

Example 3

Consider the infinite geometric series $p + p(p+1) + p(p+1)^2 + \ldots$

(a) For what values of p will the series converge?
(b) Assuming the series is convergent, calculate the sum to infinity.

Solutions

(a) In this series $a = p$ and $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = p+1$

The series will converge if $-1 < r < 1$

$\therefore -1 < p+1 < 1$

$\therefore -1-1 < p < 1-1$

$\therefore -2 < p < 0$

(b) $S_\infty = \dfrac{a}{1-r}$

$\therefore S_\infty = \dfrac{p}{1-(p+1)}$ (substitute $a = p$ and $r = p+1$)

$\therefore S_\infty = \dfrac{p}{1-p-1}$

$\therefore S_\infty = \dfrac{p}{-p} = -1$

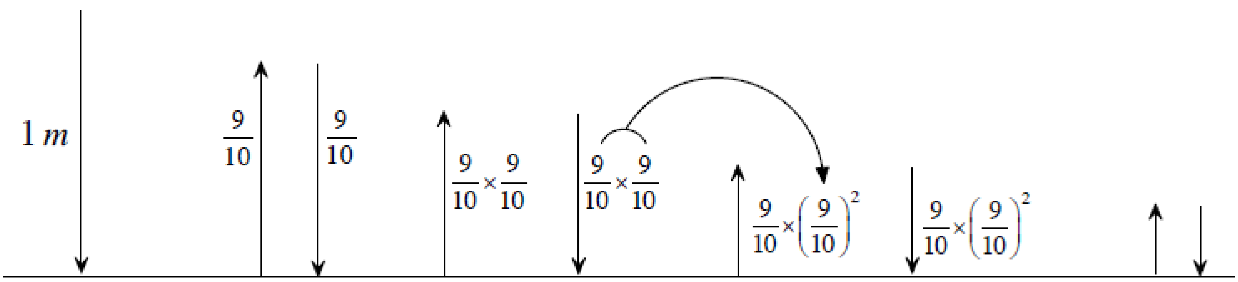
Real life applications

Example 1

A rubber ball falls from a height of one metre above the ground. It bounces back to a height of $\frac{9}{10}$ of a metre. It falls again and returns to a height of $\frac{9}{10}$ of the immediate previous height.

If this action were to continue indefinitely (theoretically), show that the total distance covered by the ball could never exceed 19 metres.

Solution



$$D = 1 + 2\left(\frac{9}{10}\right) + 2\left(\frac{9}{10}\right)^2 + 2\left(\frac{9}{10}\right)^3 + 2\left(\frac{9}{10}\right)^4 + \ldots$$

$$D = 1 + 2\left[\left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \left(\frac{9}{10}\right)^4 + \ldots\right]$$

$$D = 1 \ + \ 2\left[\frac{\left(\frac{9}{10}\right)}{1-\frac{9}{10}}\right] = 1 \ + \ 2\left[\frac{\frac{9}{10}}{\frac{1}{10}}\right] = 1 \ + 18 = 19m$$

Since the progressive sums of the terms of the series approach 19 metres, it means that it will be impossible for the distance to ever exceed 19 metres.

Summary of all formulae

Arithmetic	Geometric
$d = T_{k+1} - T_k \ \ (d = T_2 - T_1 = T_3 - T_2 = \ldots)$	$r = \frac{T_{k+1}}{T_k} \quad (r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots)$
$T_n = a + (n-1)d$	$T_n = ar^{n-1}$
$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$	$S_n = \frac{a\left(r^n - 1\right)}{r-1}$ or $S_n = \frac{a\left(1-r^n\right)}{1-r}$ where $r \neq 1$
or $S_n = \frac{n}{2}\left(a + l\right)$	$S_\infty = \frac{a}{1-r} \quad (-1 < r < 1)$ and $r \neq 0$

Misconceptions and common errors

1.	Some learners fail to distinguish between the constant ratio (r) and a constant difference(d)
2.	Some learners confuse the position (n) for the term and the value (T_n) of the term.
3.	When determining the value of n in a geometric sequence, some learners struggle to apply logarithm or exponential laws. For some they instead divide. e.g $64 = 2^n$ $n = \frac{64}{2}$ wrong Correction $n = \log_2 64$ ✓
4.	Some candidates don't know the mathematical representation of the words **least** and **most**.
5.	Some candidates don't know which formula to use in a calculation. For instance a candidate may use the general term formula yet the question may be asking for the sum of terms of a given series.
6.	Some candidates don't know the relationship between the terms of the first difference of a quadratic pattern with the terms of the quadratic pattern. In other words candidates struggle to work backwards from the second constant difference to the first differences and then to the quadratic terms.
7.	The constant difference (d) and the constant ratio (r) are sometimes calculated wrongly. i.e $d = T_1 - T_2$ wrong $r = \frac{T_1}{T_2}$ wrong The correct method is $d = T_2 - T_1$ ✓ $r = \frac{T_2}{T_1}$ ✓

Tips:

- It's important for you to know the difference between arithmetic, geometric and quadratic sequences.
- You should know which formula to use in a given question.
- Candidates should practice a lot of questions that involve the application of logs and the laws of exponents.
- Always try to visualize the question in terms of real life applications most especially when you have to apply the concept of the sum to infinity.
- Practice a lot of questions that require working backwards especially with the quadratic patterns.
- It is important for the candidate to know when the question requires to find the position of the term or the value of the term.

EXAMINATION QUESTIONS ON SEQUENCES & SERIES

Feb-March 2018

QUESTION 2

2.1 Given the following geometric sequence: $30\ ;\ 10\ ;\ \frac{10}{3}\ ;\ldots$

2.1.1 Determine n if the n^{th} term of the sequence is equal to $\frac{10}{729}$. (4)

2.1.2 Calculate: $30+10+\frac{10}{3}+\ldots$ (2)

2.2 Derive a formula for the sum of the first n terms of an arithmetic sequence if the first term of the sequence is a and the common difference is d. (4)

[10]

Feb – March 2017

QUESTION 2

Given the geometric sequence: $-\frac{1}{4}\ ;\ b\ ;\ -1\ ;\ldots\ldots$

2.1 Calculate the possible values of b. (3)

2.2 If $b=\frac{1}{2}$, calculate the 19^{th} term (T_{19}) of the sequence. (3)

2.3 If $b=\frac{1}{2}$, write the sum of the first 20 positive terms of the sequence in sigma notation. (4)

2.4 Is the geometric series formed in QUESTION 2.3 convergent? Give reasons for your answer. (2)

[12]

Nov 2018

QUESTION 2

2.1 Given the quadratic sequence: 2 ; 3 ; 10 ; 23 ; …

2.1.1 Write down the next term of the sequence. (1)

2.1.2 Determine the n^{th} term of the sequence. (4)

2.1.3 Calculate the 20^{th} term of the sequence. (2)

2.2 Given the arithmetic sequence: 35 ; 28 ; 21 ; …

Calculate which term of the sequence will have a value of -140. (3)

2.3 For which value of n will the sum of the first n terms of the arithmetic sequence in QUESTION 2.2 be equal to the n^{th} term of the quadratic sequence in QUESTION 2.1? (6)

[16]

Nov 2019

QUESTION 2

2.1 Given the quadratic sequence: 321 ; 290 ; 261 ; 234 ; ….

2.1.1 Write down the values of the next TWO terms of the sequence. (2)

2.1.2 Determine the general term of the sequence in the form $T_n = an^2 + bn + c$. (4)

2.1.3 Which term(s) of the sequence will have a value of 74? (4)

2.1.4 Which term in the sequence has the least value? (2)

2.2 Given the geometric series: $\frac{5}{8} + \frac{5}{16} + \frac{5}{32} + \ldots = K$

2.2.1 Determine the value of K if the series has 21 terms. (3)

2.2.2 Determine the largest value of n for which $T_n > \frac{5}{8192}$ (4)

[19]

Feb-March 2018
QUESTION 3

The first three terms of an arithmetic sequence are -1 ; 2 and 5.

3.1 Determine the n^{th} term, T_n, of the sequence. (2)

3.2 Calculate T_{43}. (2)

3.3 Evaluate $\sum_{k=1}^{n} T_k$ in terms of n. (3)

3.4 A quadratic sequence, with general term T_n, has the following properties:

- $T_{11} = 125$
- $T_n - T_{n-1} = 3n - 4$

Determine the first term of the sequence. (6)
[13]

Feb-March 2017
QUESTION 3

3.1 6 ; 6 ; 9 ; 15 ; ... are the first four terms of a quadratic number pattern.

3.1.1 Write down the value of the fifth term (T_5) of the pattern. (1)

3.1.2 Determine a formula to represent the general term of the pattern. (4)

3.1.3 Which term of the pattern has a value of 3 249? (4)

3.2 Determine the value(s) of x in the interval $x \in [0^\circ ; 90^\circ]$ for which the sequence -1 ; $2\sin 3x$; 5 ; ..... will be arithmetic. (4)
[13]

Nov 2018
QUESTION 3

A geometric series has a constant ratio of $\frac{1}{2}$ and a sum to infinity of 6.

3.1 Calculate the first term of the series. (2)

3.2 Calculate the 8^{th} term of the series. (2)

3.3 Given: $\sum_{k=1}^{n} 3(2)^{1-k} = 5{,}8125$ Calculate the value of n. (4)

3.4 If $\sum_{k=1}^{20} 3(2)^{1-k} = p$, write down $\sum_{k=1}^{20} 24(2)^{-k}$ in terms of p. (3)
[11]

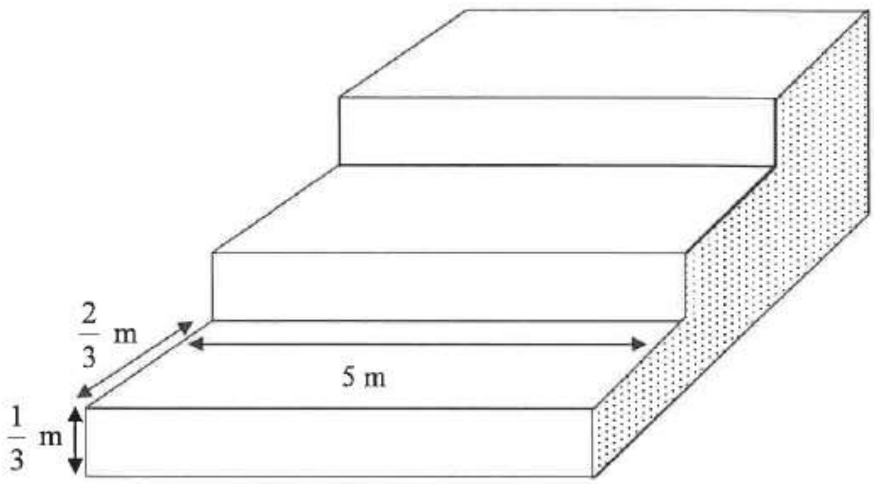
Nov 2019

QUESTION 3

3.1 Without using a calculator, determine the value of: $\sum_{y=3}^{10} \frac{1}{y-2} - \sum_{y=3}^{10} \frac{1}{y-1}$ (3)

3.2 A steel pavilion at a sports ground comprises of a series of 12 steps, of which the first 3 are shown in the diagram below.

Each step is 5 m wide. Each step has a rise of $\frac{1}{3}$ m and has a tread of $\frac{2}{3}$ m, as shown in the diagram below.

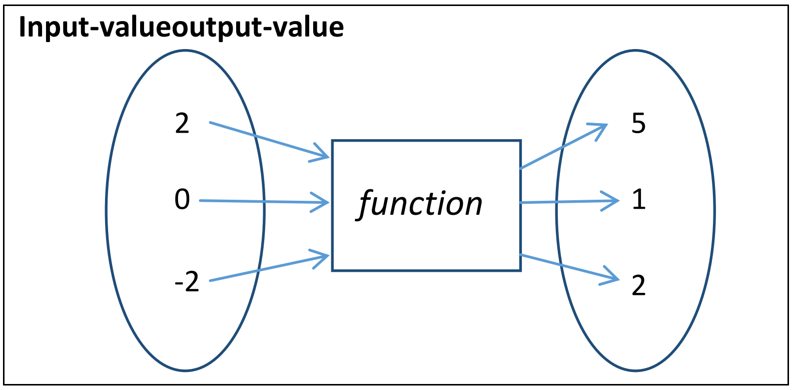


The open side (shaded on sketch) on each side of the pavilion must be covered with metal sheeting. Calculate the area (in m^2) of metal sheeting needed to cover both open sides. (6)

[9]

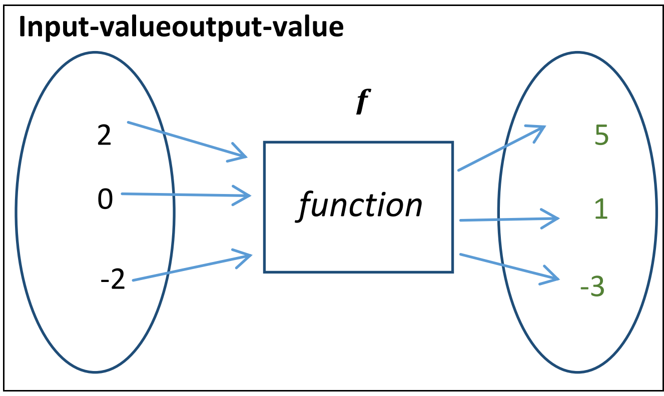
SECTION 3
FUNCTIONS & INVERSES

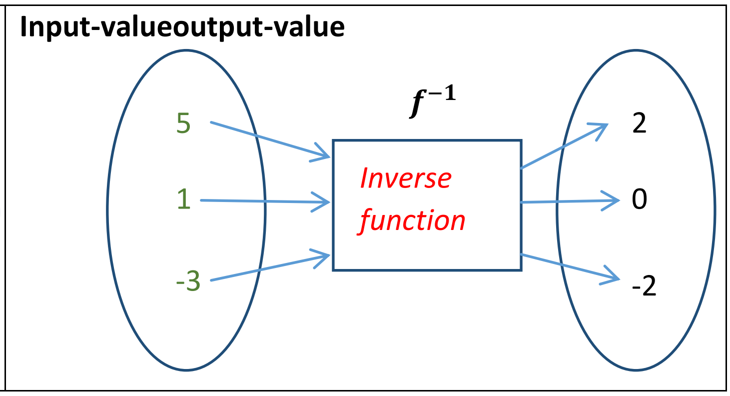
Function is a relationship or a rule between the input (x-values)/(Domain) and the output (y-values)/ (Range)



Inverse function is a rule that reverses the input and output values of a function.

If $\boldsymbol{f}$ represents a function , then $\boldsymbol{f^{-1}}$ is the inverse function.





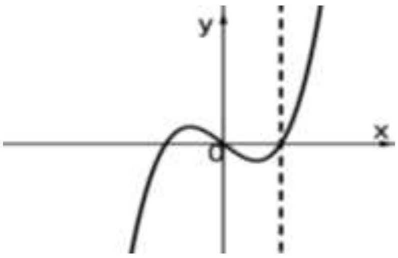
Functions can be ***one- to – one*** or ***many – to – one*** Relations.

NOTE: if a relation is one- to – many, then it is **NOT** a function.

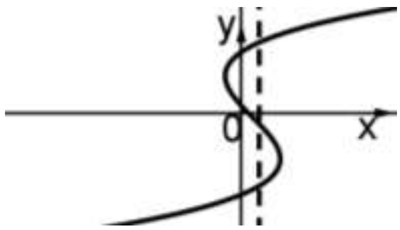
HOW TO DETERMINE WHETHER THE GRAPH IS A FUNCTION OR NOT

i. Vertical –line test:

The **vertical –line test** is used to determine whether a graph is a function or not a function. To determine whether a graph is a function, draw a vertical line parallel to the *y*-axis or perpendicular to the *x*- axis. If the line intersects the graph once then graph is a function. If the line intersects the graph more than once then the relation is not a function of *x*. Because functions are single-valued relations and a particular *x*-value is mapped onto one and only one *y*-value.



Function



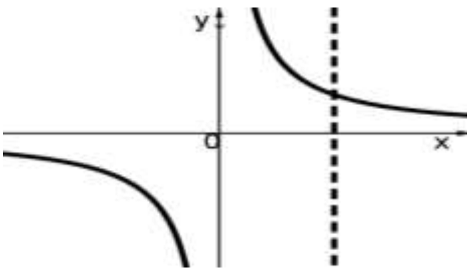
not a function (one to many relation)

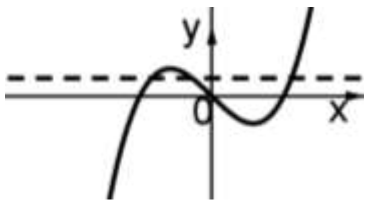
TEST FOR ONE –TO- ONE FUNCTION

ii. Horizontal- line test

The **horizontal –line test** is used to determine whether a function is one-to-one function.

To determine whether a graph is one –to -one function, draw a horizontal line parallel to the *x*-axis or perpendicular to the *y*- axis. If the line intersects the graph once the graph is one - to- one function. If the line intersects the graph more than once then the relation is **not a** one-to-one function.





many-to-one function

one-to-one function

EXERCISE 1 **50 marks**

1. State whether the following relations are functions or not. If the graph is a function, state whether the function is one-to-one or many-to-one.

1.1 $\{(-2;-7)\,;(0;-1)\,;(1;2)\,;(2;6)\}$

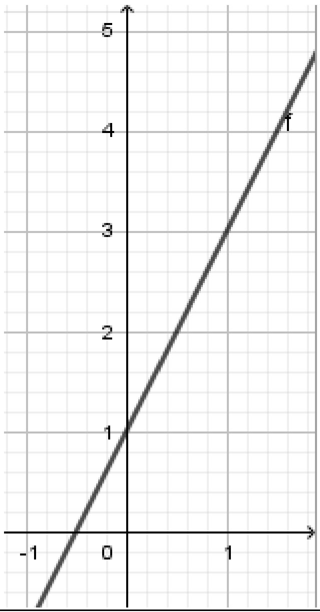
1.2 $\{(-2;6);(-1;3);(0;2)\,;(1;3)\,;(2;6)\}$

1.3 $\{(-2;16);(4;1)\,;(4;6);(3;7)\}$

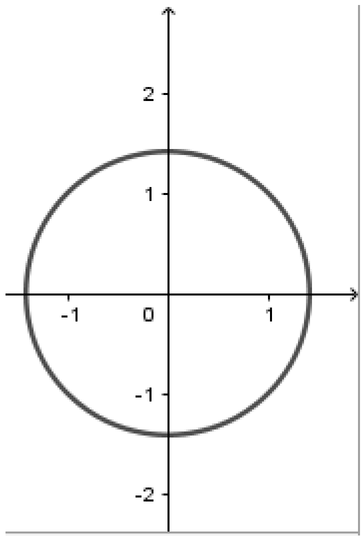
1.4

x	y
-2	5
-1	2
0	1
1	2
2	5

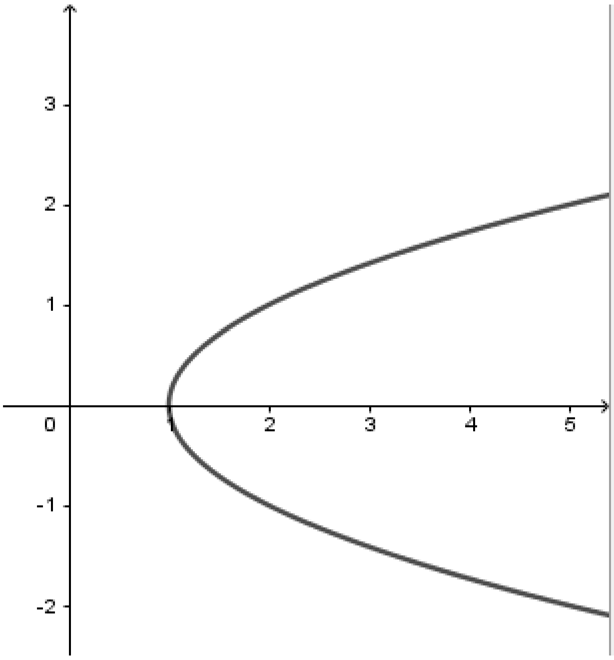
1.5



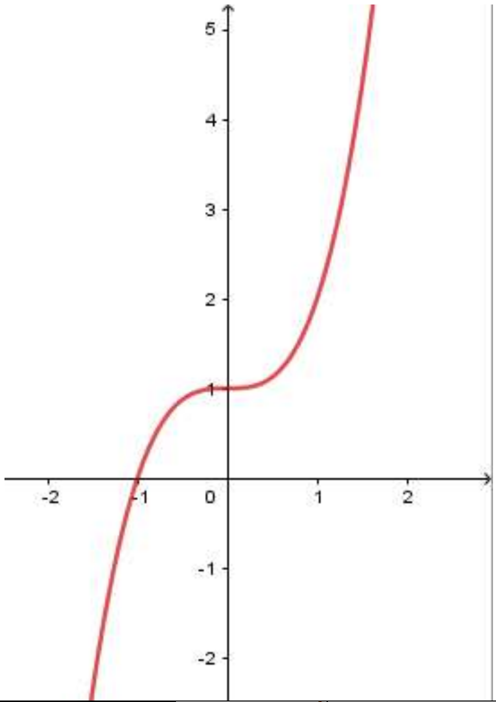
1.6



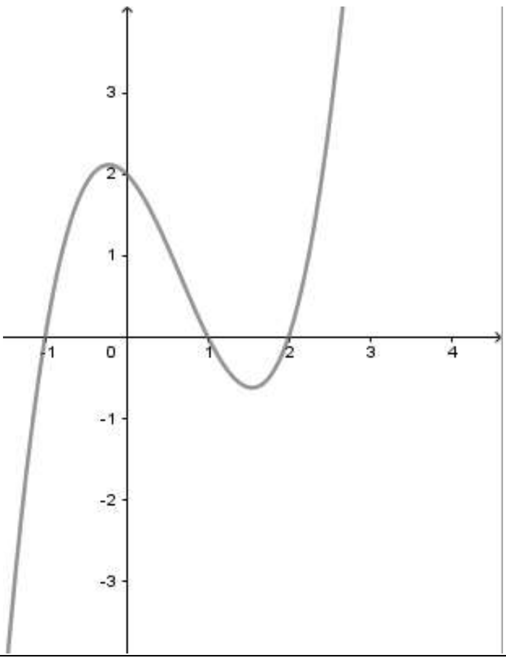
1.7



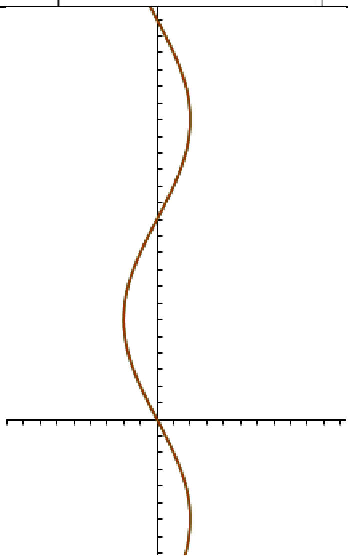
1.8



1.9



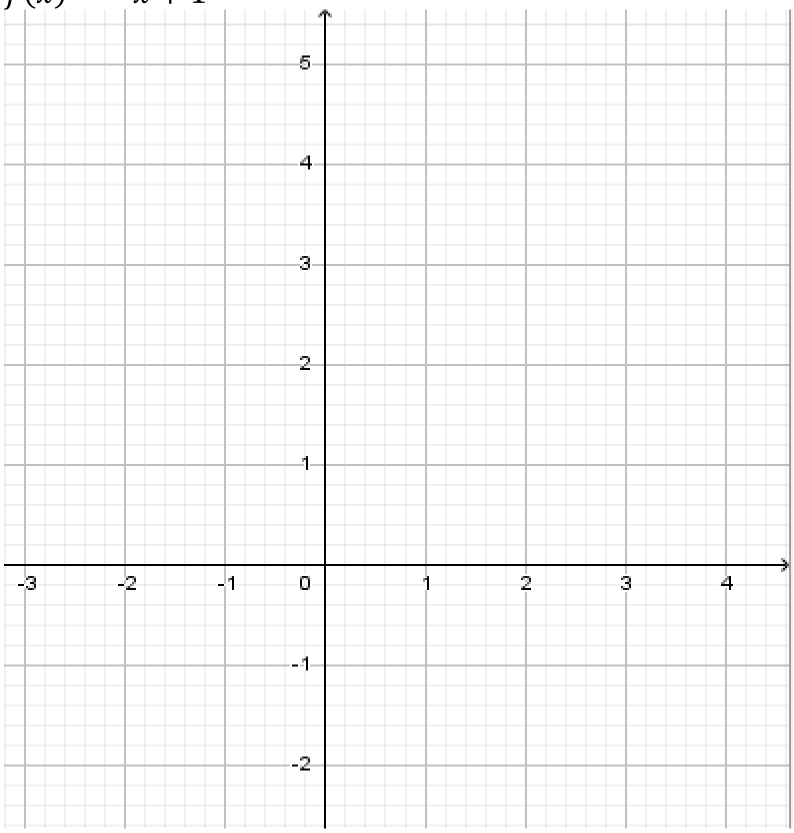
1.10



(10)

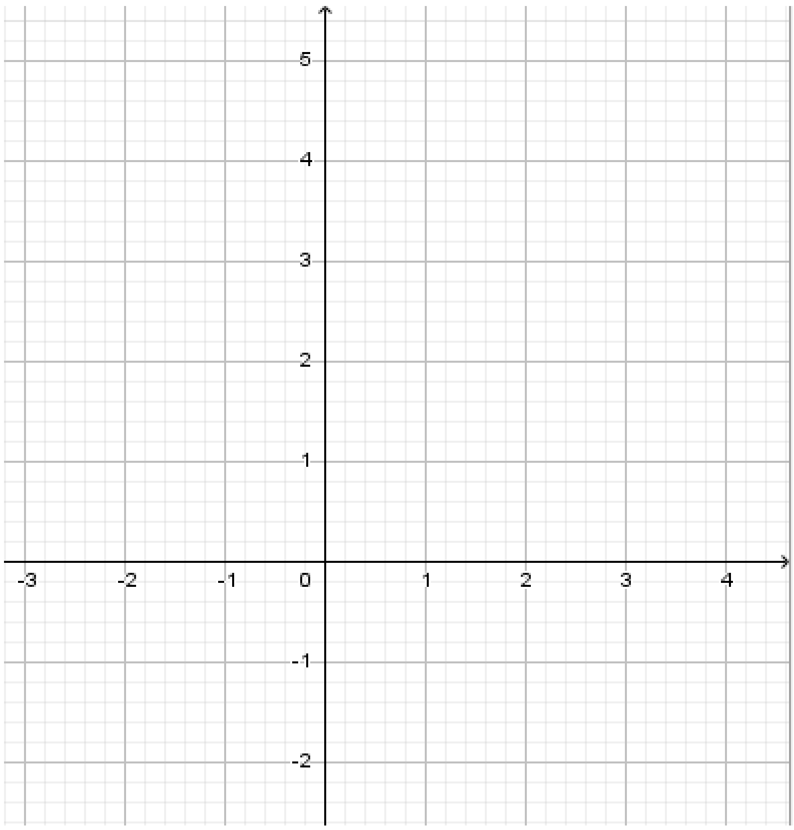
2. Sketch the following functions:

2.1 $f(x) = -x + 1$



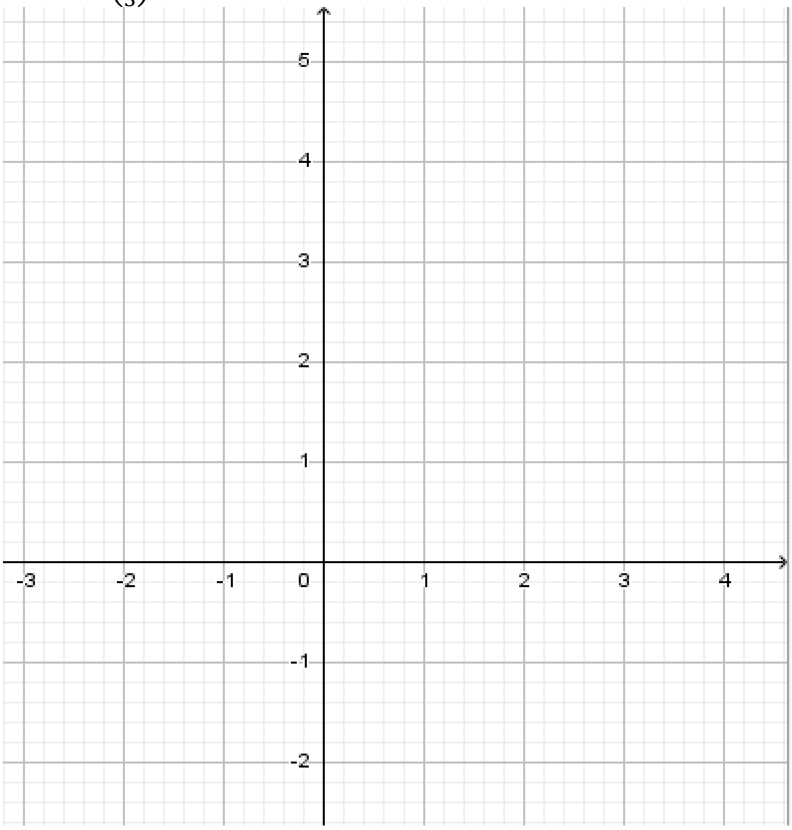
2

2.2 $g(x) = -x^2 + 4$



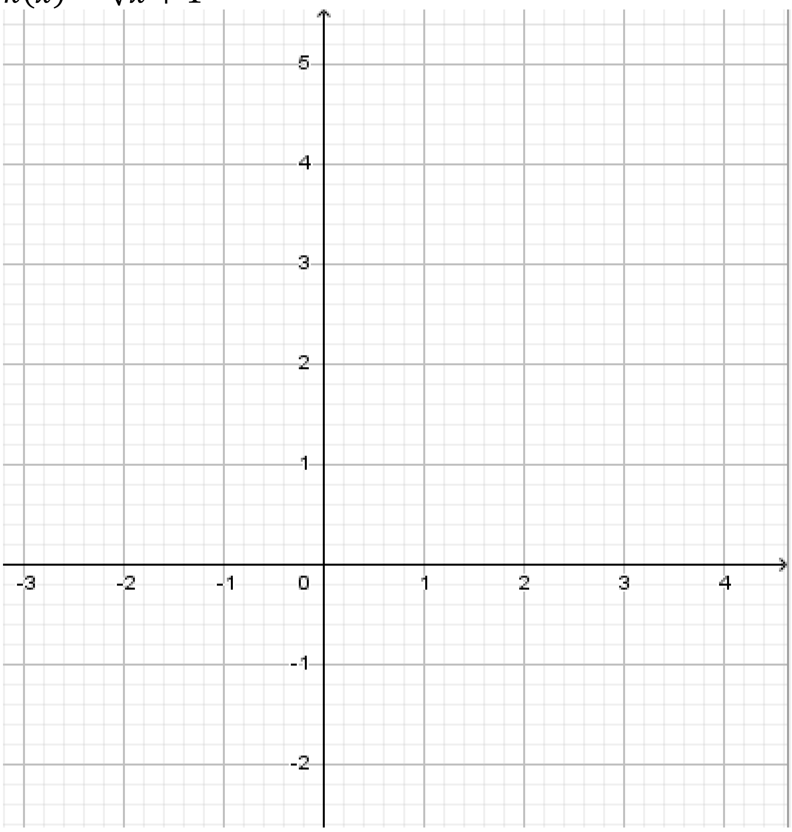
4

2.3 $h(x) = \left(\frac{1}{3}\right)^x - 1$



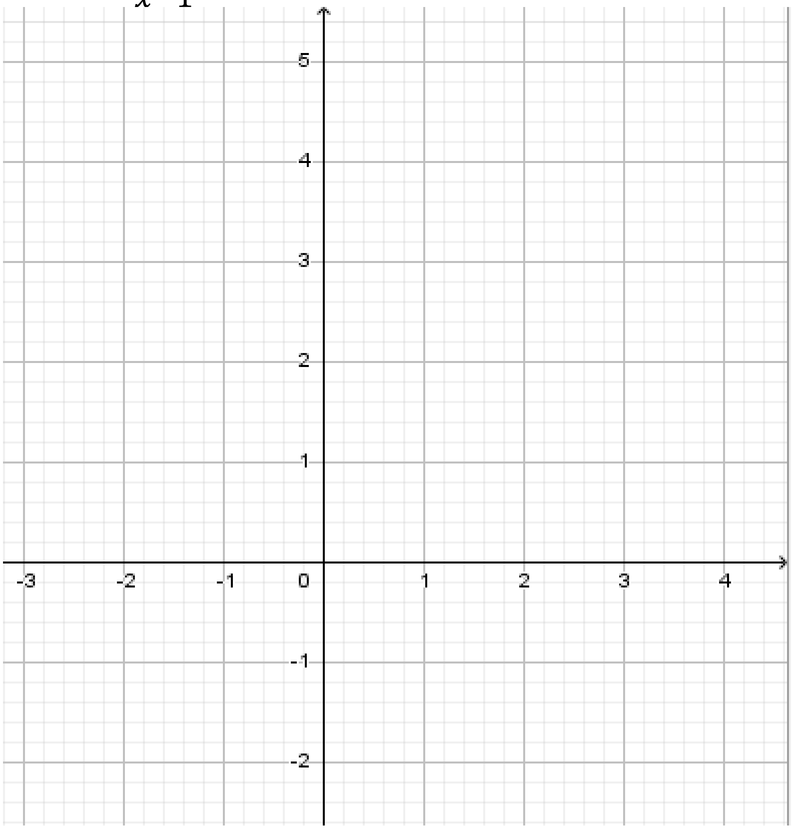
4

2.4 $k(x) = \sqrt{x + 1}$



3

2.5 $j(x) = \frac{2}{x-1} + 2$



4

(17)

3. For the graphs sketched in **question 2 above**, state the domain and Range

$f(x) = -x + 1$	DOMAIN	RANGE
$g(x) = -x^2 + 4$	DOMAIN	RANGE
$h(x) = \left(\frac{1}{3}\right)^x - 1$	DOMAIN	RANGE
$k(x) = \sqrt{x+1}$	DOMAIN	RANGE
$j(x) = \frac{2}{x-1} + 2$	DOMAIN	RANGE

(10)

4. For each of the functions given in **question 2 above,** write the equation of the new function formed after a translation of 1 $unit$ right and 2 $units$ down.

$f'(x) =$
$g'(x) =$
$h'(x) =$
$k'(x) =$
$j'(x) =$

(10)

5. Explain why $h(x)$ is a function and state with a reason(s) why it is a $1-to-1$ function.(3)

TOTAL: 50

METHOD ON HOW TO DETERMINE THE EQUATION OF THE INVERSE

First interchange/swap *x* and *y*, then make y the subject of the formula

Example1: **Linear function**

Determine the inverse of $f(x) = 2x + 3$

Solution

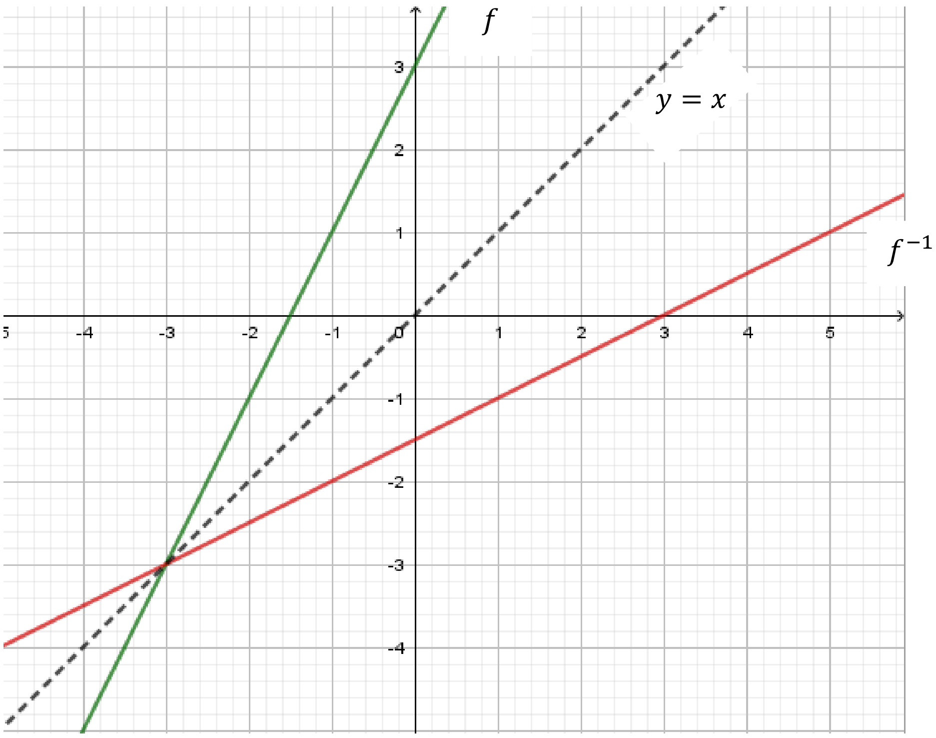
$y = 2x + 3$

$x = 2y + 3$ Interchange ***x***and***y*** this is also the inverse but is in the form $x = \cdots$

$$x - 3 = 2y$$

$$\frac{x-3}{2} = y \text{this is in the form} f^{-1}(x) = \frac{x-3}{2} \text{or } y = \cdots$$

Sketch of $f(x) = 2x + 3$and$f^{-1}(x) = \frac{x-3}{2}$ on the same set of axes.



Both f and f^{-1} intersect at a point $(-3; -3)$. The line $y = x$ is the axis of symmetry.

	Domain	Range
$f(x)$	$x \in \mathbb{R}$	$y \in \mathbb{R}$
$f^{-1}(x)$	$x \in \mathbb{R}$	$y \in \mathbb{R}$

- Both f and f^{-1} have the same domain and range but the y- intercept of f is now thex- intercept of f^{-1}.
- $f(x)$and$f^{-1}(x)$ are both one- to – one functions.

Example 2: **Quadratic function**

Determine the inverse of $f(x) = 2x^2$

Solution:

$$y = 2x^2$$

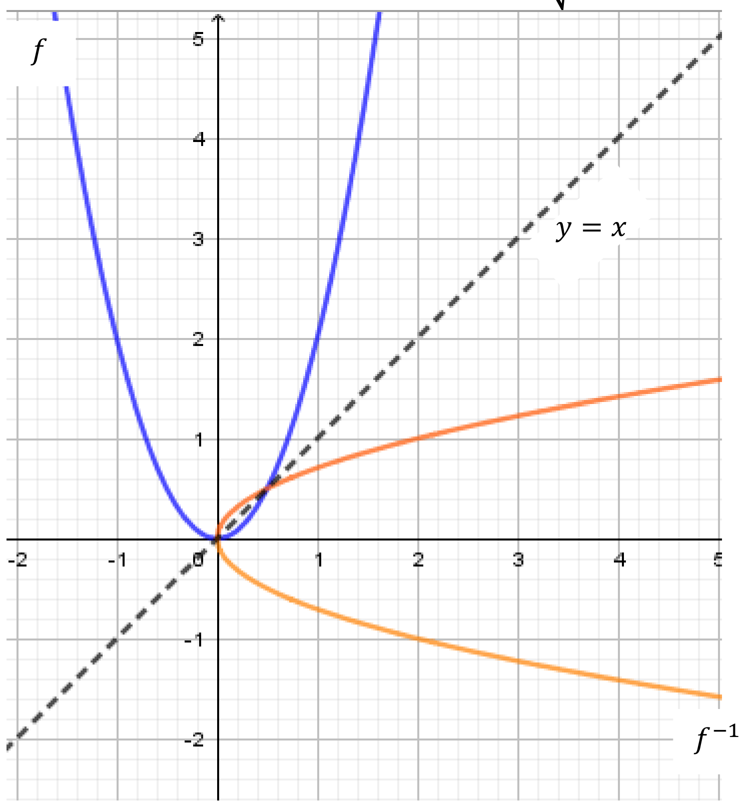
$x = 2y^2$ Interchange ***x*** and ***y***; this is also the inverse but is in the form $x = \cdots$

$$\frac{x}{2} = y^2$$

$$\pm\sqrt{\frac{x}{2}} = y$$

$\pm\sqrt{\frac{x}{2}} = y$ this is in the form $f^{-1}(x) = \pm\sqrt{\frac{x}{2}}$ or $y = \cdots$

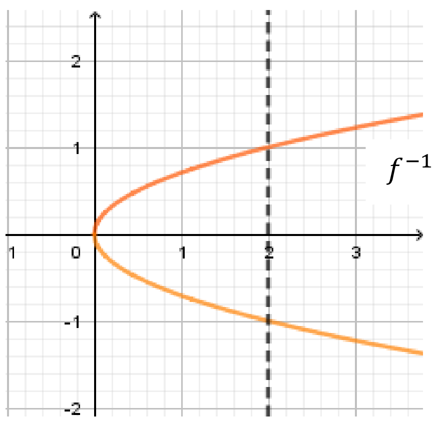
Sketch of $f(x) = 2x^2$ and $f^{-1}(x) = \pm\sqrt{\frac{x}{2}}$ on the same set of axes.



Both f and f^{-1} intersect at two points. The line $y = x$ is the axis of symmetry.

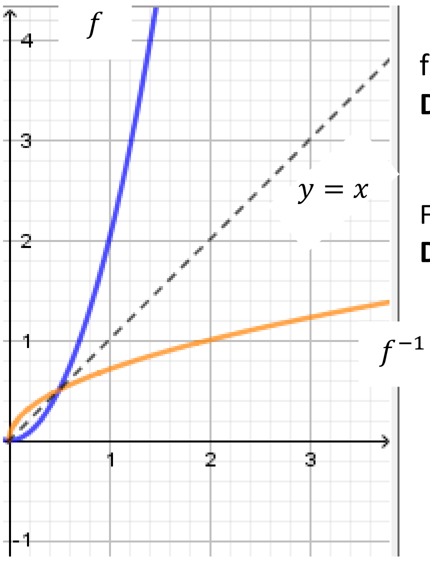
	Domain	Range
$f(x)$	$x \in \mathbb{R}$	$y \geq 0$
$f^{-1}(x)$	$x \geq 0$	$y \in \mathbb{R}$

- From the sketch above, the domain and range of $f(x)$ have interchanged forming range and domain respectively of $f^{-1}(x)$
- But $f^{-1}(x)$ the inverse of $f(x)$ is **NOT** a function because according to the vertical line test the graph of $f^{-1}(x)$ is cut twice by the vertical line.
- $f^{-1}(x)$ is a one- to- many relation.



But if the domain of $f(x)$is restricted to $x \geq 0$ or $x \leq 0$ then the inverse will also be a function.

Restriction 1



for $f(x)$

Domain: **Range:**

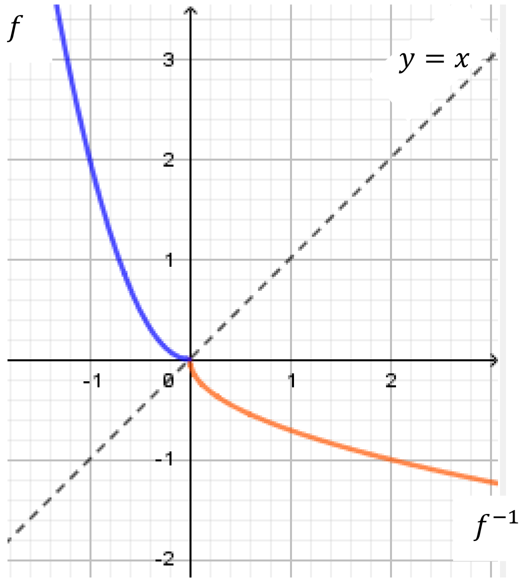
$x \geq 0 y \geq 0$

For $f^{-1}(x)$

Domain: **Range:**

$x \geq 0 y \geq 0$

Restriction 2



for $f(x)$

Domain: **Range:**

$x \geq 0 y \geq 0$

For $f^{-1}(x)$

Domain: **Range:**

$x \geq 0 y \geq 0$

Example 3 **Exponential function**

Determine the inverse of $f(x) = 2^x$

Solution:

$$y = 2^x$$

$x = 2^y$ Interchange ***x*** and ***y*** this is also the inverse but is in the form $x = \cdots$

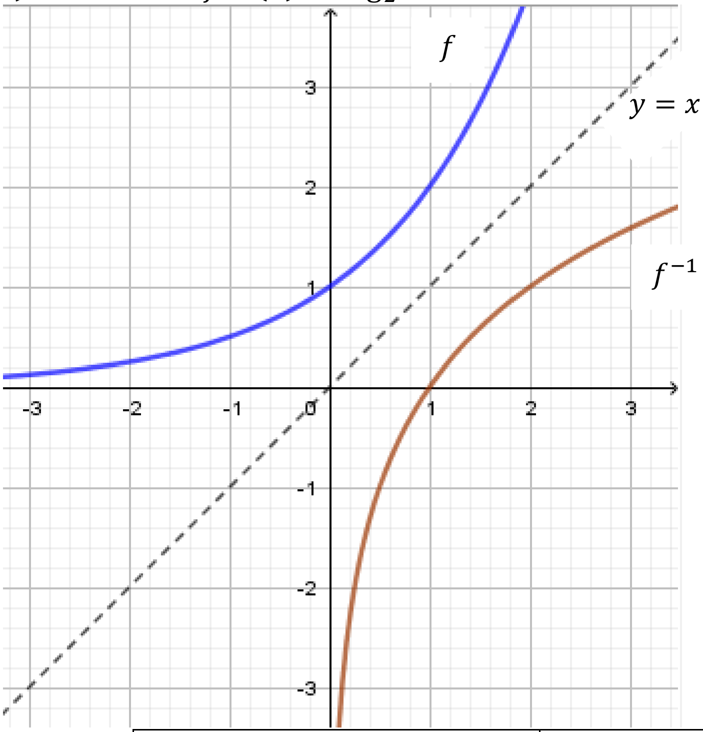
$\log_2 x = \log_2 2^y$ introduce logarithm to the base of 2 on both sides of the equation

$\log_2 x = y\log_2 2$ but $\log_a a = 1$ $\Rightarrow \log_2 2 = 1$

$$\log_2 x = y$$

$\log_2 x = y$ this is in the form $f^{-1}(x) = \log_2 x$ or $y = \cdots$

Sketch of $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$ on the same set of axes.



	Domain	Range
$f(x)$	$x \in \mathbb{R}$	$y \geq 0$
$f^{-1}(x)$	$x \geq 0$	$y \in \mathbb{R}$

Both f and f^{-1} are one-to-one functions. The line $y = x$ is the axis of symmetry.

Example 4: **Exponential function**

Determine the inverse of $f(x) = \left(\frac{1}{2}\right)^x$

Solution:

$y = \left(\frac{1}{2}\right)^x$ $x = 2^y$ Interchange ***x*** and ***y*** this is also the inverse but is in the form $x = \cdots$

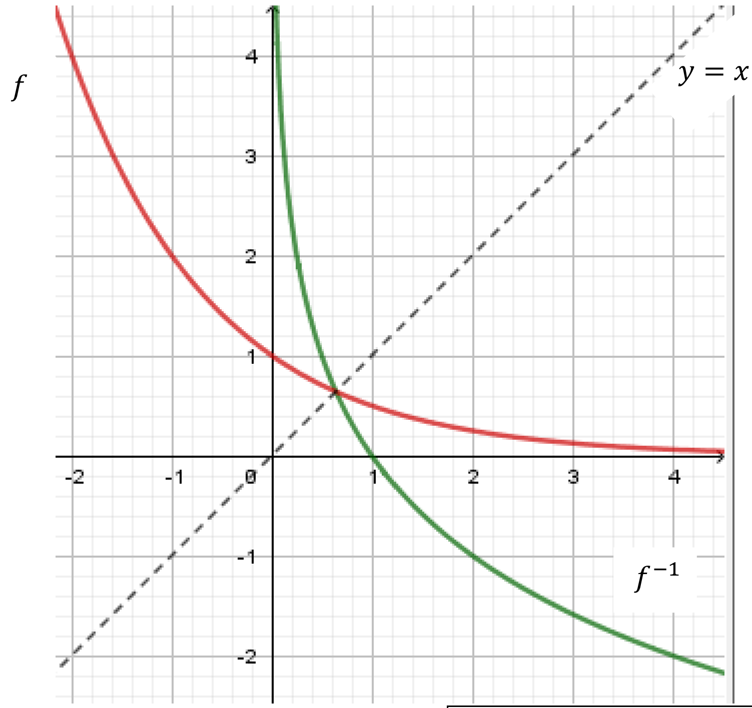
$\log_{\frac{1}{2}} x = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^y$ introduce logarithm to the base of $\frac{1}{2}$ on both sides of the equation

$\log_{\frac{1}{2}} x = y\log_{\frac{1}{2}} \frac{1}{2}$ but $\log_a a = 1$ $\Rightarrow \log_{\frac{1}{2}} \frac{1}{2} = 1$

$$\log_{\frac{1}{2}} x = y$$

$\log_{\frac{1}{2}} x = y$ this is in the form $f^{-1}(x) = \log_{\frac{1}{2}} x$ or $y = \cdots$

Sketch of $f(x) = \left(\frac{1}{2}\right)^x$ and $f^{-1}(x) = \log_{\frac{1}{2}} x$ on the same set of axes.



	Domain	Range
$f(x)$	$x \in \mathbb{R}$	$y \geq 0$
$f^{-1}(x)$	$x \geq 0$	$y \in \mathbb{R}$

Both f and f^{-1} are one-to-one functions. The line $y = x$ is the axis of symmetry.

EXERCISE 2

1. Determine the inverse for each of the functions below: **(2 marks each)**

   1.1 $f(x) = \frac{1}{2}x - 3$

   1.2 $j(x) = -2x^2 + 2$

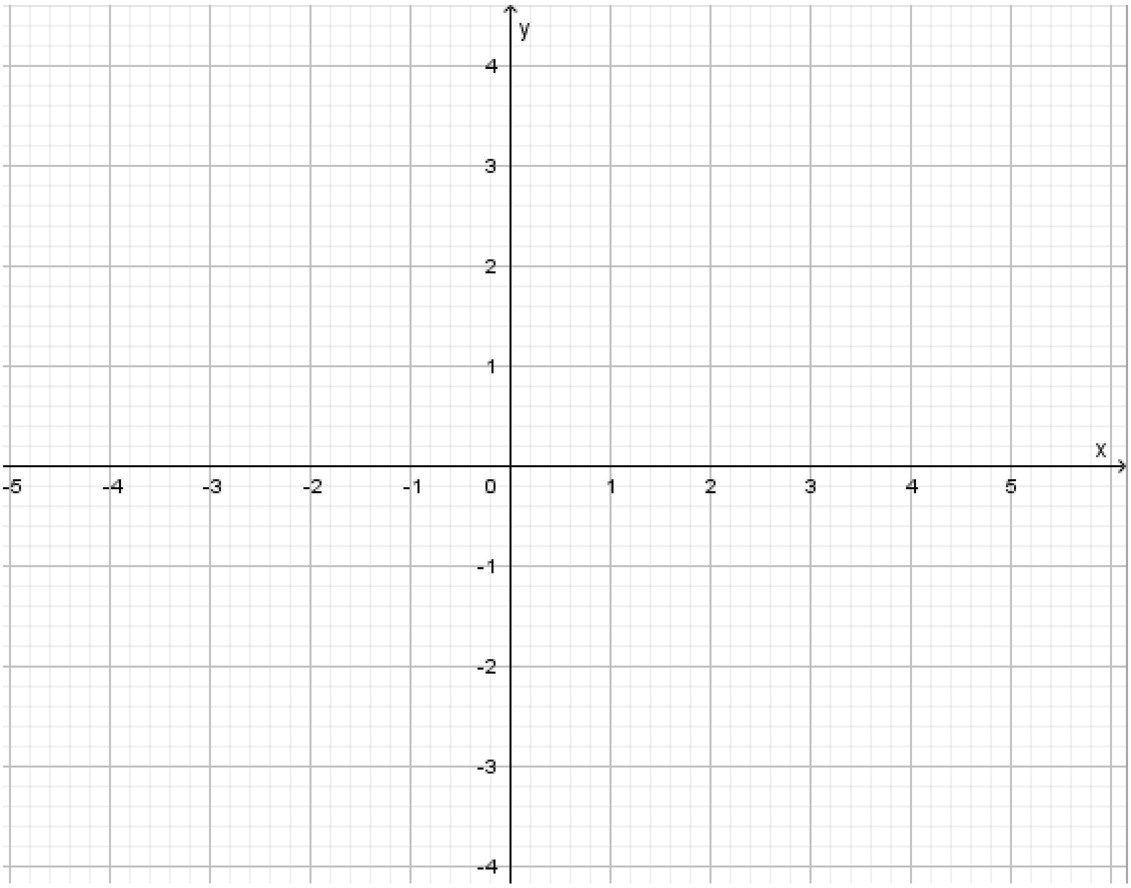
   1.3 $m(x) = 2^{-x} - 2$

   1.4 $n(x) = \frac{1}{x+1} - 2$

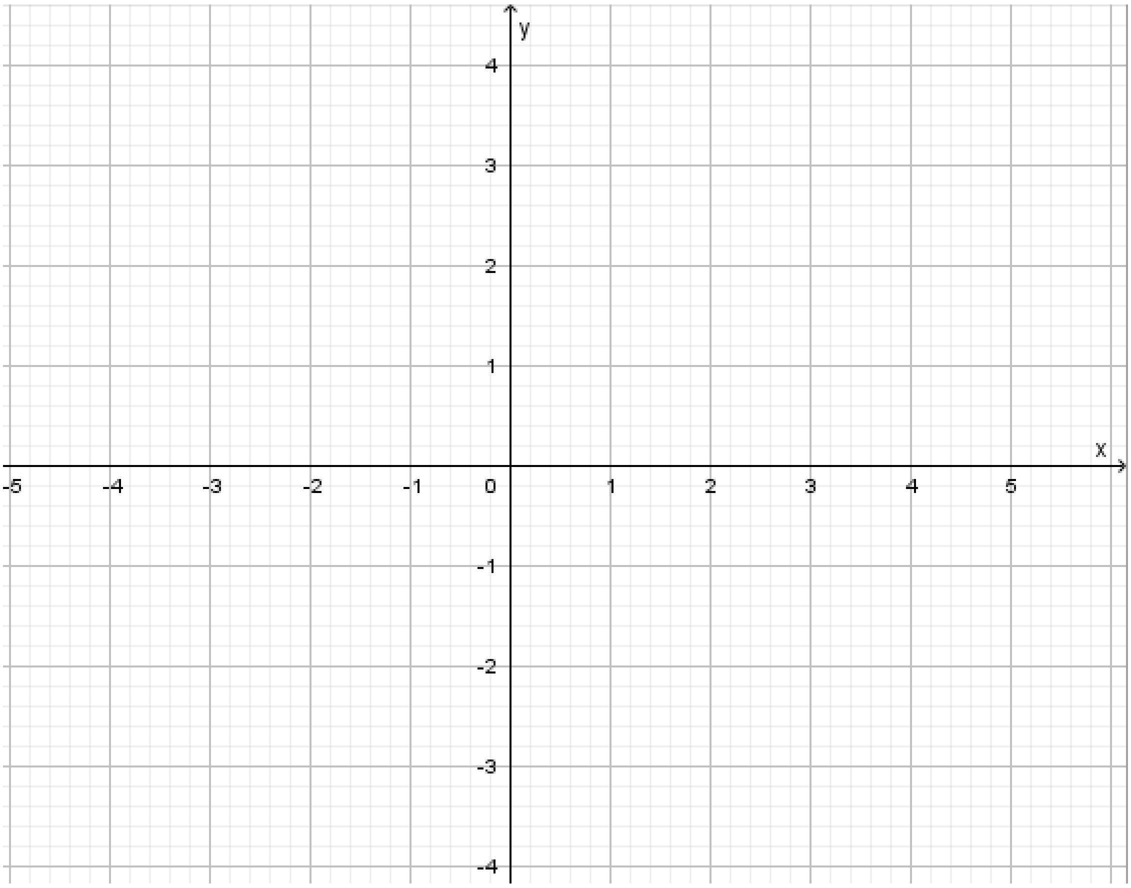
(8)

2. For each of the functions in **Question 1** above, sketch both the function and the inverse on the same set of axes. Clearly show the asymptotes where necessary. **(4 marks each function/4 marks each inverse)**

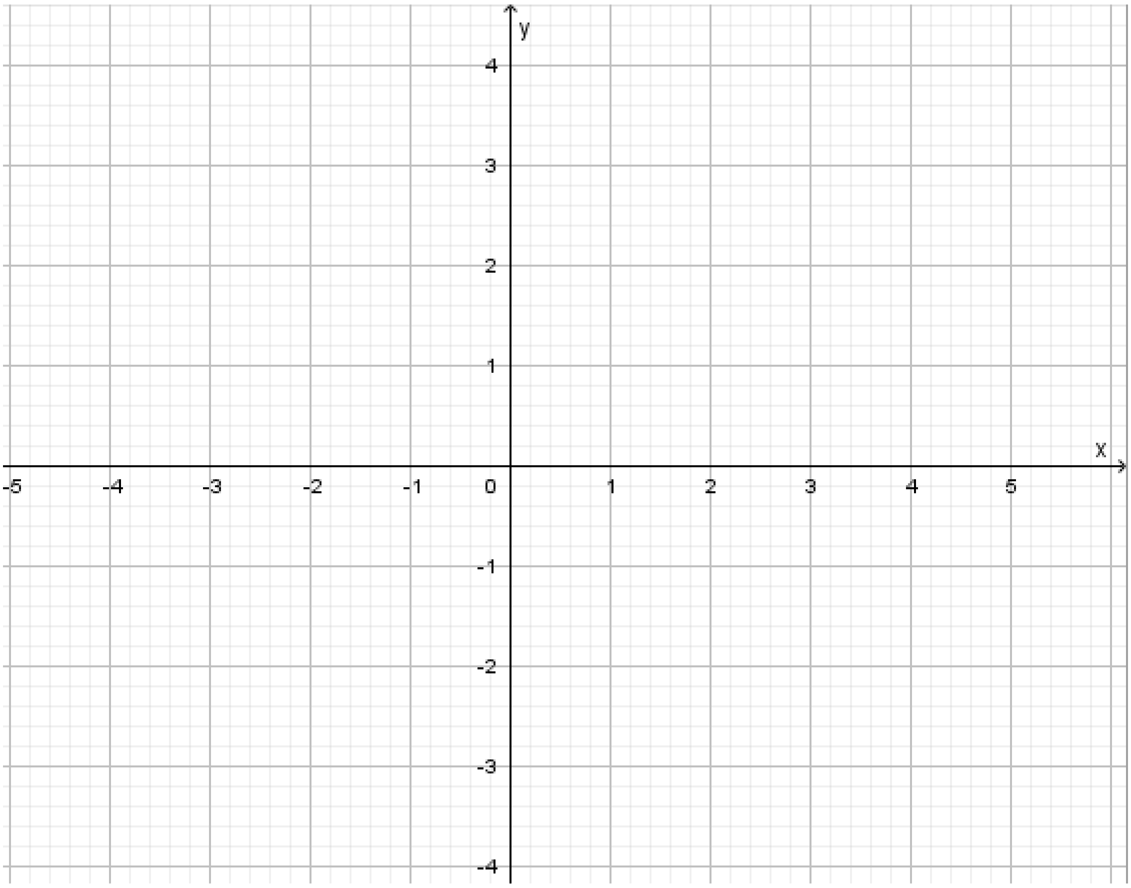
2.1



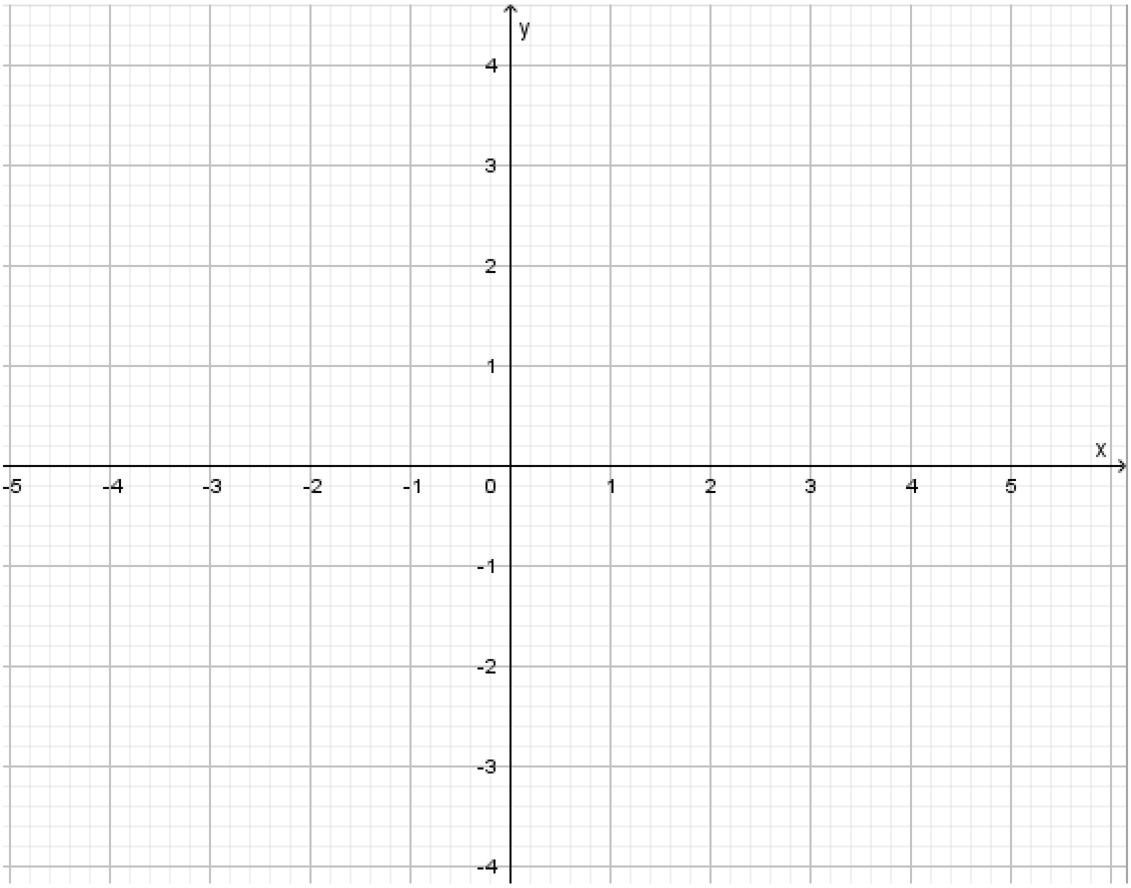
2.2



2.3



2.4



(32)

3. State the domain and range for each of the functions and their inverses sketched in **Question 2** above. **(2 marks each function/ 2 marks each inverse)**

	Function		**Inverse**	
3.1	Domain:	Range:	Domain:	Range:
3.2	Domain:	Range:	Domain:	Range:
3.3	Domain:	Range:	Domain:	Range:
3.4	Domain:	Range:	Domain:	Range:

(16)

TOTAL: 56

Misconceptions and common errors

1.	Some candidates are unable to recall the formula to calculate the x-coordinate at the turning point of a parabola. They use $x = \frac{-2b}{a}$ wrong instead of $x = \frac{-b}{2a}$.
2.	If given the hyperbolic function, some candidates fail to change the sign when determining the equation of the vertical asymptote. E.g $f(x) = \frac{2}{x+1} + 3$ $y = 3$ ✓ $x = 1$ wrong Correction $x = -1$✓
3.	Many candidates fail to write the correct notations for the domain and the range.
4.	When sketching graphs, candidates fail to draw the correct graph for the given function. **For example** , someone draws a sketch of an exponential graph instead of a hyperbola. OR a candidate draws the required sketch graph but with wrong intercepts.
5.	For $f(x) = ax^2 + bx + c$; $g(x) = \frac{a}{x+p} + q$, some candidates don't know how to use the sign of the value of a as guide for the correct sketch. Yet if they understood the significance of a, it would be easy for them to draw the right sketches.
6.	When determining the inverse of a function, some candidates don't know or struggle to make y the subject of the formula after interchanging x and y. this makes them lose some marks.
7.	Learners sometimes fail to write the correct restriction(s) for the domain of the parabola such that it's inverse is a function. i.e $x > 0$ wrong or $x < 0$ wrong Corrections $x \geq 0$✓ or $x \leq 0$✓
8.	Many candidates struggle to write a new equation formed after a graph undergoes vertical and horizontal shifts.

Tips:

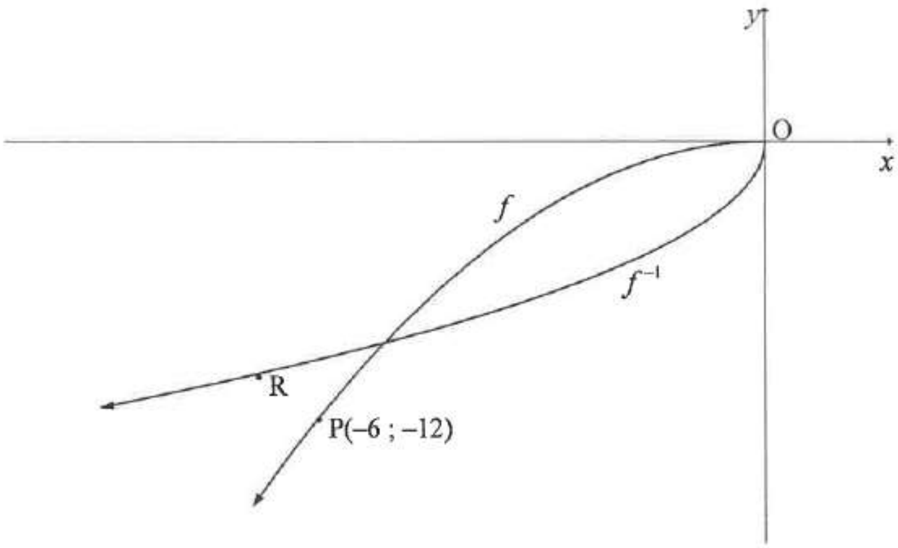
- Candidates must regularly revise the grade 10 and 11 basic concepts on functions.
- Candidates should always remember the vertical line test and horinzontal line test.
- When drawing a sketch graph, always know the shape of the graph based on the value of (a) given, calculate the intercepts, asymptotes and the turning points accordingly.
- It is important that a candidate can read off the solutions from the graph.
- Translations (vertical and horizontal shifts), a candidate should know how moving the graph horizonatlly or vertically affects the equation of the graph.
- A candidate should be able to write the axis of symmetry of a graph. For the parabola the axis of symmetry is given by $x = p$; ***where*** $p \in \mathbb{R}$ and the hyperbola has two axes of symmetry.i.e $y = x + c$ & $y = -x + c$
- Candidates should practice more questions the involve graphical interpretations. Ie questions like ; for which values of x if
  i. $f(x) > 0$
  ii. $f(x).g(x) \leq 0$

EXAMINATION QUESTIONS ON FUNCTIONS & INVERSES

Nov 2018

QUESTION 4

In the diagram below, the graph of $f(x) = ax^2$ is drawn in the interval $x \leq 0$.
The graph of f^{-1} is also drawn. P(–6 ; –12) is a point on f and R is a point on f^{-1}.



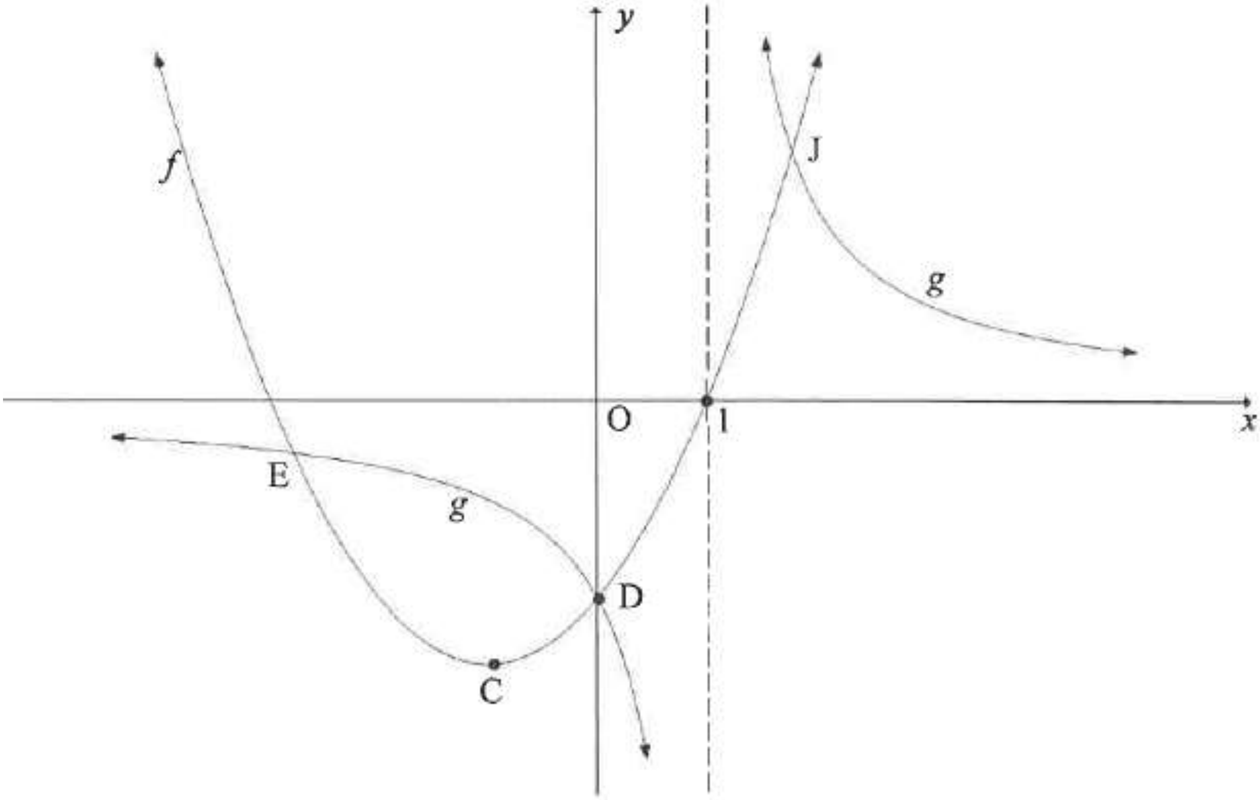
4.1	Is f^{-1} a function? Motivate your answer.	(2)
4.2	If R is the reflection of P in the line $y = x$, write down the coordinates of R.	(1)
4.3	Calculate the value of a.	(2)
4.4	Write down the equation of f^{-1} in the form $y = ...$	(3)
		[8]

Nov 2019

QUESTION 4

Below are the graphs of $f(x) = x^2 + bx - 3$ and $g(x) = \frac{a}{x+p}$.

- f has a turning point at C and passes through the x-axis at $(1 ; 0)$.
- D is the y-intercept of both f and g. The graphs f and g also intersect each other at E and J.
- The vertical asymptote of g passes through the x-intercept of f.



4.1	Write down the value of p.	(1)
4.2	Show that $a = 3$ and $b = 2$.	(3)
4.3	Calculate the coordinates of C.	(4)
4.4	Write down the range of f.	(2)
4.5	Determine the equation of the line through C that makes an angle of $45°$ with the positive x-axis. Write your answer in the form $y = \ldots$	(3)
4.6	Is the straight line, determined in QUESTION 4.5, a tangent to f? Explain your answer.	(2)
4.7	The function $h(x) = f(m - x) + q$ has only one x-intercept at $x = 0$. Determine the values of m and q.	(4)

[19]

Nov 2018

QUESTION 5

Given: $f(x)=\dfrac{-1}{x-1}$

5.1 Write down the domain of f. (1)

5.2 Write down the asymptotes of f. (2)

5.3 Sketch the graph of f, clearly showing all intercepts with the axes and any asymptotes. (3)

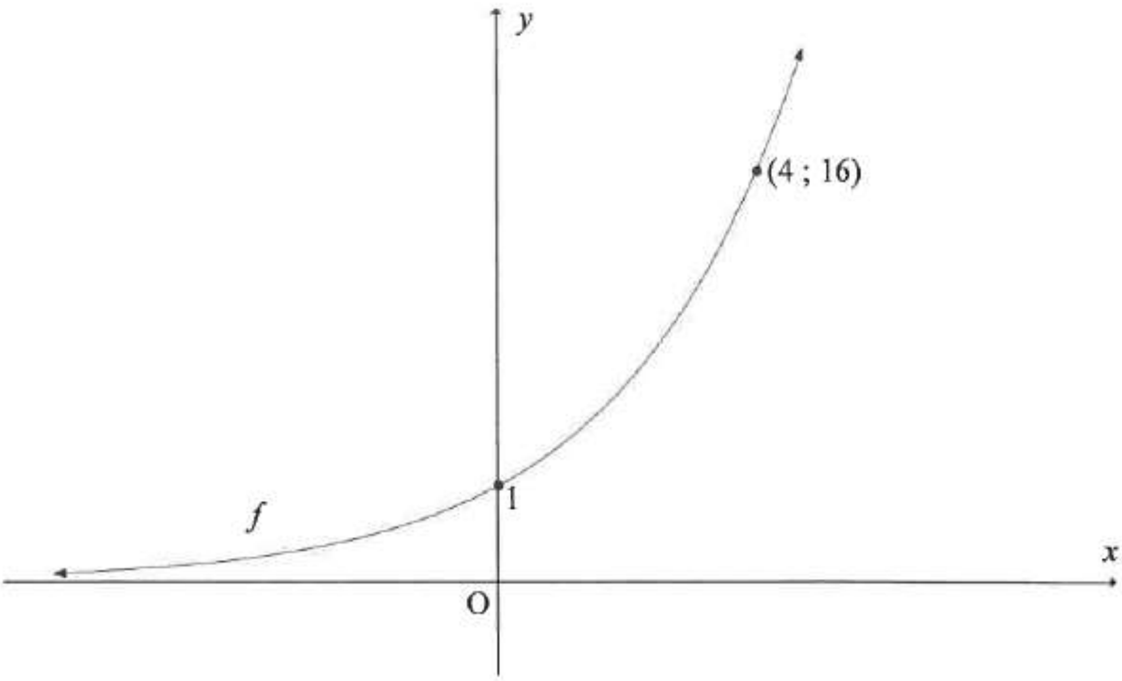
5.4 For which values of x will $x.f^{/}(x)\geq 0$? (2)

[8]

Nov 2019

QUESTION 5

Sketched below is the graph of $f(x) = k^x$; $k > 0$. The point $(4 ; 16)$ lies on f.



5.1 Determine the value of k. (2)

5.2 Graph g is obtained by reflecting graph f about the line $y = x$. Determine the equation of g in the form $y = \ldots$ (2)

5.3 Sketch the graph g. Indicate on your graph the coordinates of two points on g. (4)

5.4 Use your graph to determine the value(s) of x for which:

5.4.1 $f(x) \times g(x) > 0$ (2)

5.4.2 $g(x) \leq -1$ (2)

5.5 If $h(x) = f(-x)$, calculate the value of x for which $f(x) - h(x) = \frac{15}{4}$ (4)

[16]

SECTION 4
FINANCIAL MATHEMATICS

Growth:

simple interest: $A = P(1 + in)$ or

compound interest: $A = P(1 + i)^n$

Decay:

simple decay (called the straight line method): $A = P(1 - in)$

compound decay (called the reducing balance method): $A = P(1 - i)^n$

- If the annual interest that is quoted (the nominal interest rate) is compoundedmore frequently than once a year, the effective interest rate will be higher than thenominal interest rate, and is determined using the formula:

$$1 + i_{eff} = \left(1 + \frac{i_m}{m}\right)^m$$

- To determine the amount accumulated after an investment has been growing withcompound interest that is compounded k times per year: divide the quoted interestrate by k and multiply the number of years by k.
- When more than one transaction occurs, draw a time-line to visualise what hashappened over time. Remember to take all values to any ONE moment in time,before adding or subtracting values.
  Use the logic that 'total of money in = total ofmoney out'.
  When taking values back in time, you are finding the Pvalue for a known Avalue,so the formula becomes $P = A(1 + i)^{-n}$

EXERCISE 1

1. Portia buys a motor car for R150 000. The car depreciates at a rate of 12% **p.a**. What is the car worth after 5 years if depreciation is calculatedusing:
   1.1 the straight-line method.
   1.2 the reducing-balance method.
2. Zweli invests R12 000 in a savings account which pays 9% per annumcompounded monthly. Calculate the value of his investment in ten years'time.

3. Sarah deposits R15 500 into an account. The interest rate for the first two years is 10% p.a. compoundedquarterly. It then changes to 8,5% p.a., compoundedmonthly for the next 3 years, and then to 11% p.a.,compounded semi-annually thereafter. How muchmoney will she have in her account after 7 years?
4. Nomsi deposits R20 000 into an account at an interest rate of 9% p.a. compounded quarterly.Three years later she deposits another amount of Rx.Two years after that the interest rate changes to 10% p.a., compounded annually. She withdraws R12 000at the time that the interest rate changes. At theend of 8 years she has R43 062,27 in her account.
   Determine the value of *x*.

5. Philemon takes out a loan of R 50 000 torenovate his house. Interest on the loan is 9% p.a.compounded monthly for the first two years, and then changes to 9,5% p.a.,compounded half yearly. He makes a payment of R20 000 one year after takingout the loan, and another payment of R 25 000 two years later. How much willhe still owe on his loan four years after taking out the loan?

INVESTMENTS OR LOANS INVOLVING ANNUITIES

Definition: An annuity is a series of equal investment payments or loan repayments at regular intervals subject to a rate of interest over a period of time.

FUTURE VALUE ANNUITIES

In a **future value annuity**, money is invested at regular intervals in order to save money for the future.

The magic of compound interest makes the investment grow in value, especially if the interest rate is above the current inflation rate.

Typical future value annuities include retirement annuities, in which people save money each month so as to receive pension payments once they retire.

$$F = \frac{x[(1+i)^n - 1]}{i}$$

where:

x equal payments made per period

$i$interest rate

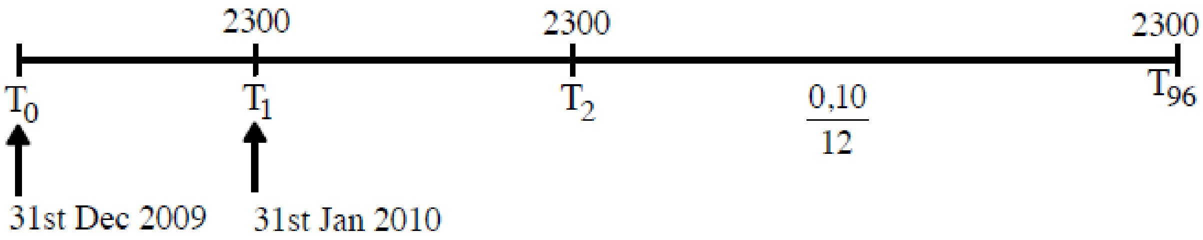
$n$number of payments made

Example 1

Patrick decided to start saving money for a period of eight years starting on 31st December 2009. At the end of January 2010 (in one month's time), he deposited an amount of R2 300 into the savings plan. Thereafter, he continued making deposits of R2 300 at the end of each month for the planned eight year period. The interest rate remained fixed at 10% per annum compounded monthly. How much will he have saved at the end of his eight year plan which started on the 31st December 2009?

Solution

In this example, the duration of the loan is 8 years (96 months). However, the number of payments is 96 because of the first payment being made one month after the starting of the savings plan.



There are a total of 96 payments and the duration of the investment is 8 years (96 months).

The future value can now be calculated using the formula:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore F = \frac{2300\left[\left(1+\frac{0,10}{12}\right)^{96} - 1\right]}{\frac{0,10}{12}} = R336\ 216,47$$

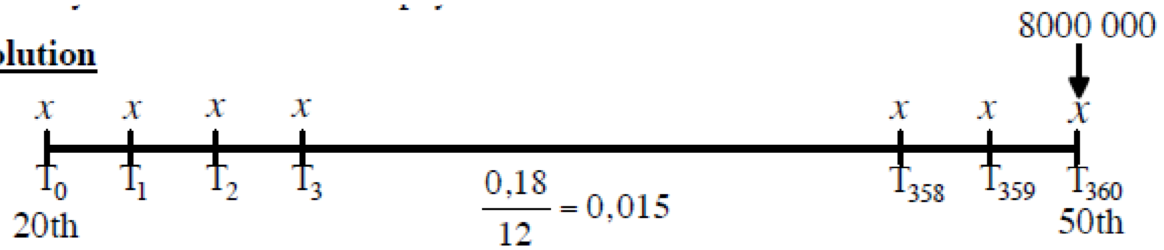
Payment made immediately

When a payment is made immediately into the account, the total number of payments increases by 1 **i.e** $n \rightarrow n+1$

Example 2

Dayna has just turned 20 years old and has a dream of saving R8 000 000 by the time she reaches the age of 50. She starts to pay equal monthly amounts into a retirement annuity which pays 18% per annum compounded monthly. Her first payment starts on her 20th birthday and her last payment is made on her 50th birthday. How much will she pay each month?

Solution



The savings is for a period of 30 years $\times 12$ months $= 360$. The number of payments of x will be 361 (there is a payment at T_0). The future value (F) in this example is R8 000 000.

$$\therefore 8\,000\,000 = \frac{x\left[(1{,}015)^{361}-1\right]}{0{,}015}$$

$$\therefore 8\,000\,000 \times 0{,}015 = x\left[(1{,}015)^{361}-1\right]$$

$$\therefore \frac{8\,000\,000 \times 0{,}015}{\left[(1{,}015)^{361}-1\right]} = x$$

$$\therefore x = \text{R}558{,}41$$

Annuity payments made in advance

It is often the case that in annuity investments, payments are made **in advance**. This means that the last payment in the annuity is made **one month before the investment is paid out**. The next example deals with this type of annuity.

Example 3

In order to supplement his state pension after retirement, a school teacher aged 30 takes out a retirement annuity. He makes monthly payments of R1 000 into the fund and the payments start immediately. The payments are made **in advance**, which means that the last payment of R1 000 is made one month before the annuity pays out. The interest rate for the annuity is 12% per annum compounded monthly.

Calculate the future value of the annuity in twenty-five years' time.

$$\frac{0,12}{12}=0,01$$

A total of 300 payments will be made.

The value of the annuity will first be calculated at T_{299}. The accumulated amount at T_{299} will then be grown for one month.

$$F=\frac{1000\left[(1,01)^{300}-1\right]}{0,01}$$

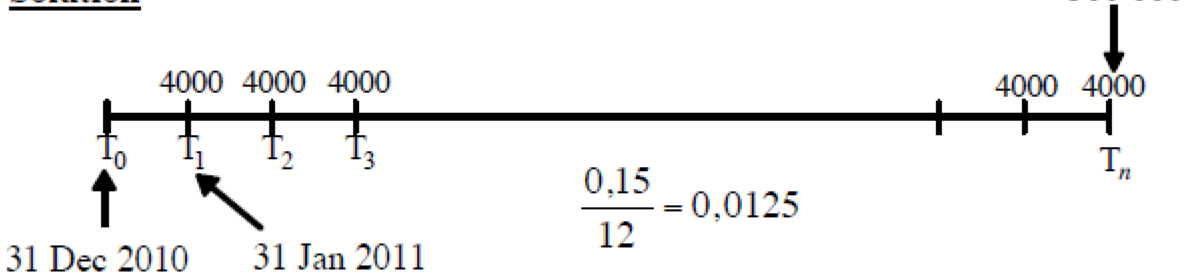
$$\therefore F=1\ 878\ 846,626$$

$$A=1\ 878\ 846,626(1,01)^{1}=R1\ 897\ 635,09$$

Example 4

It is the 31st December 2010. Anna decides to start saving money and wants to save R300 000 by paying monthly amounts of R4000, starting in one month's time (on 31st January 2011), into a savings account paying 15% per annum compounded monthly. How many payments of R4000 will be made? The duration of the savings starts on the 31st December 2010, even though the first payment is not made on the 31st December 2010.

Solution



$$\frac{0,15}{12}=0,0125$$

There are n number of payments of R4000.

$$300\ 000 = \frac{4000\left[(1{,}0125)^n - 1\right]}{0{,}0125}$$

$$\therefore \frac{300\ 000 \times 0{,}0125}{4000} = (1{,}0125)^n - 1$$

$$\therefore \frac{300\ 000 \times 0{,}0125}{4000} + 1 = (1{,}0125)^n$$

$$\therefore 1{,}9375 = (1{,}0125)^n$$

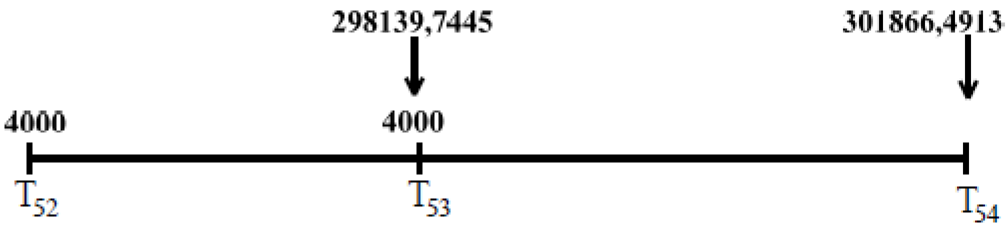
$$\therefore (1{,}0125)^n = 1{,}9375$$

$$\therefore n = \log_{1{,}0125} 1{,}9375$$

$$\therefore n = 53{,}24189314$$

The value of n represents the number of payments made. This means that Anna will make 53 payments of R4 000, but what does the decimal represent?

Let's explore this on a time-line.



The amount accumulated after 53 months is:

$$\frac{4000\left[(1{,}0125)^{53} - 1\right]}{0{,}0125} = 298139{,}7445$$

This amount is clearly less than the R300 000 required.

The amount accumulated after 54 months is:

$$\frac{4000\left[(1{,}0125)^{53} - 1\right]}{0{,}0125}(1{,}0125)^1 = 301866{,}4913$$

This amount is clearly more than the R300 000 required.

Therefore, if 53 payments of R4 000 are made and the accumulated amount is left to grow for a few days (or weeks) into the next month, the amount of R300 000 will be acquired. At the end of the 54th month, the investor will have saved more than R300 000. Therefore there will be 53 payments of R4 000 and the accumulated amount will need to grow into the next month in order for the R300 000 to be obtained.

PRESENT VALUE ANNUITIES

In a **present value annuity**, a sum of money is normally borrowed from a financial institution and paid back with interest by means of regular payments at equal intervals over a time period. The loan is said to be amortised (paid off) when it together with interest charges is paid off. The interest is calculated on the reducing balance.

Formula

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

where:

x equal payments made per period

$i$interest rate

$n$number of payments made

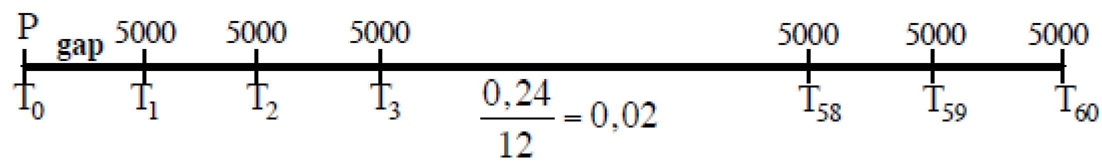
Note:

The formula for P can only be used if there is a **gap** between the loan and the first payment. If the payments are monthly, then there must be a one month gap between the loan and the first payment.

Example 1

Malibongwe takes out a bank loan to pay for his new car. He repays the loan by means of monthly payments of R5 000 for a period of five years starting one month after the granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the purchase price of his new car.

Solution



There are 60 months in five years and the number of payments are 60.

$$P = \frac{5000\left[1-(1,02)^{-60}\right]}{0,02}$$

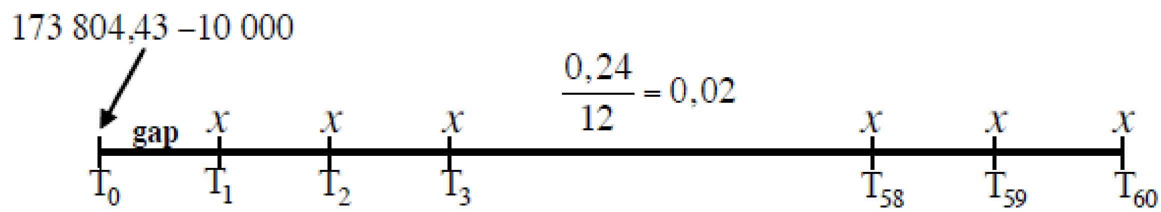
$$\therefore P = R173\ 804,43$$

Note:
If a loan is taken out and a payment is made at the same time, then this payment must be subtracted from the original loan. This payment is really a deposit and must be deducted from the loan before applying the present value annuity formula.

Example 2 (when the loan repayment is made immediately)

Malibongwe takes out a bank loan to pay for his new car. He pays an **initial amount**(deposit) of R10 000. He then makes monthly payments for a period of five years starting one month after the granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the monthly payments if the car originally cost him R173 804,43.

Solution



$$173\ 804{,}43 - 10\ 000 = \frac{x\left[1-(1{,}02)^{-60}\right]}{0{,}02}$$

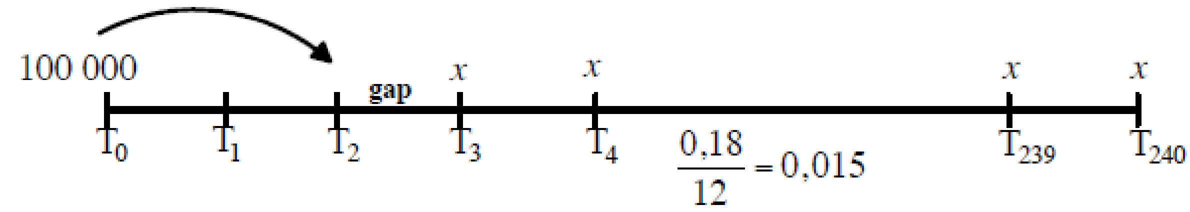
$$\therefore \frac{163\ 804{,}43 \times 0{,}02}{\left[1-(1{,}02)^{-60}\right]} = x$$

$$\therefore x = \text{R}4\ 712{,}32$$

Example 3 (Delayed loan repayment)

Melanie takes out a twenty year loan of R100 000. She repays the loan by means of equal monthly payments starting **three months** after the granting of the loan. The interest rate is 18% per annum compounded monthly. Calculate the monthly payments.

Solution



The present value formula only works if there is a gap between the loan and the first payment. Therefore, it is necessary to first grow the loan toT_2, which is a gap before the first payment. The number of payments in this deferred annuity will therefore only be 238, because two are missing (at T_1 and T_2).

$$100\,000(1{,}015)^2 = \frac{x\left[1-(1{,}015)^{-238}\right]}{0{,}015}$$

$$\therefore 100\,000(1{,}015)^2 \times 0{,}015 = x\left[1-(1{,}015)^{-238}\right]$$

$$\therefore \frac{100\,000(1{,}015)^2 \times 0{,}015}{\left[1-(1{,}015)^{-238}\right]} = x$$

$$\therefore x = \text{R}1591{,}35$$

Example 4

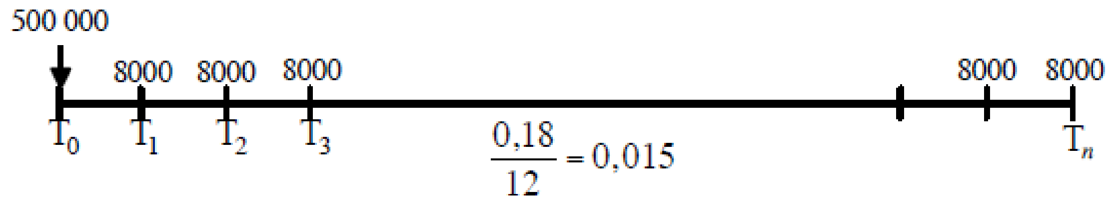
Peter borrows R500 000 from a bank and repays the loan by means of monthly payments of R8 000, starting one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly.

(a) How many payments of R8 000 will be made and what will the final lesser payment be?

(b) How long is the savings period?

Solutions

(a)



$$500\ 000 = \frac{8000\left[1-(1{,}015)^{-n}\right]}{0{,}015}$$
$$\therefore \frac{500\ 000 \times 0{,}015}{8000} = 1-(1{,}015)^{-n}$$
$$\therefore (1{,}015)^{-n} = 1-\frac{500\ 000 \times 0{,}015}{8000}$$

$$\therefore (1{,}015)^{-n} = 0{,}0625$$
$$\therefore -n = \log_{1{,}015} 0{,}0625$$
$$\therefore -n = -186{,}2221025$$
$$\therefore n = 186{,}2221025$$

There will be 186 payments of R8 000 into the annuity. The decimal here indicates that there will be a **final payment** which is less than R8 000.
The final payment, call it *x*, can be calculated as follows:

$$500\ 000 = \frac{8000\left[1-(1{,}015)^{-186}\right]}{0{,}015} + x(1{,}015)^{-187}$$
$$\therefore 500\ 000 - \frac{8000\left[1-(1{,}015)^{-186}\right]}{0{,}015} = x(1{,}015)^{-187}$$
$$\therefore 110{,}4090631 = x(1{,}015)^{-187}$$
$$\therefore \frac{110{,}4090631}{(1{,}015)^{-187}} = x$$
$$\therefore x = \text{R}1787{,}12$$

THE BALANCE OUTSTANDING ON A LOAN AT A GIVEN TIME

It is sometimes useful to calculate the balance still owed on a loan at a given time during the course of the loan.

$$B = \frac{x[1-(1+i)^{-n}]}{i}$$

where:

x equal payments made per period

$i$interest rate

$n$number outstanding of payments

OR

Grow the loan for the number of payments made and subtract the future value of the payments made.

$$B \ = P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$$

where:

x equal payments made per period

$i$interest rate

$n$number of payments made

NOTE:

$B = \frac{x[1-(1+i)^{-n}]}{i}$ can only be used if the periodic repayments remain unchanged through out the entire loan period. Once the repayment changes then use

$B \ = P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$

Example

James takes out a one year bank loan of R18 000 to pay for an expensive laptop.

The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan.

Calculate his balance outstanding after he has paid the sixth instalment.

Solutions

$$B = \frac{1650,24\left[1-(1,015)^{-6}\right]}{0,015}$$

$$\therefore B = R9401,73$$

OR

Grow the loan to T_6. Then determine the future value of the payments at T_6. Subtract to obtain the balance outstanding.

$$B = 18\,000(1{,}015)^6 - \frac{1650{,}24\left[(1{,}015)^6 - 1\right]}{0{,}015} = 9401{,}72$$

(The answer will be slightly different to the answer using the previous method).

SINKING FUNDS

Many businesses will purchase equipment which will be used for a given period of time. After a number of years, this equipment is usually sold at scrap value and new upgraded equipment is bought. The business will often set up a savings plan at the time of purchasing the original equipment. This savings plan is a future value annuity which is called a **sinking fund** in the world of business.

Misconceptions and common errors

1.	Candidates tend to confuse the formula of compound interest $A = P(1+i)^n$ and that compound decay/ Reducing Balance Method $A = P(1-i)^n$. Some candidates don't know when to apply the correct formula.
2.	Some candidates don't know how to find the value of n after substitution into the formula $A = P(1+i)^n$
3.	When solving questions with annuities, some candidates may use the Present Value formula instead of the Future Value formula and **vice vasa**.
4.	When a person buys a house and he/she pays a deposit and then takes out a loan, some candidates don't subtract the deposit already paid. They end up using the original cost of the house as their present value. This becomes a wrong substitution.
5.	Calculator error. Many candidates enter incorrect values or wrong operation commands into the calculator and this affects the accuracy of the value from the calculator.

Tips:

- The candidate should identify how the interest is compounded. i.e interest compounded monthly, compunded quarterly , compounded semi-annually.
- Its important for the candidate to choose the relevant formulae for the question.
- Candidates should do more practice using a calculator in order to minimize the calculator errors due to poor use of a calculator.
- Apart from annuities, candidates should also practice questions involving the use of timelines.
- Candidate should double check the calculator value by re-entering the calculator commands.

EXAMINATION QUESTIONS ON FINANCIAL MATHEMATICS

Nov 2018

QUESTION 7

7.1 Selby decided today that he will save R15 000 per quarter over the next four years. He will make the first deposit into a savings account in three months' time and he will make his last deposit at the end of four years from now.

7.1.1 How much will Selby have at the end of four years if interest is earned at 8,8% per annum, compounded quarterly? (3)

7.1.2 If Selby decides to withdraw R100 000 from the account at the end of three years from now, how much will he have in the account at the end of four years from now? (3)

7.2 Tshepo takes out a home loan over 20 years to buy a house that costs R1 500 000.

7.2.1 Calculate the monthly instalment if interest is charged at 10,5% p.a., compounded monthly. (4)

7.2.2 Calculate the outstanding balance immediately after the 144th payment was made. (5)

[15]

Nov 2019

QUESTION 6

6.1 Two friends, Kuda and Thabo, each want to invest R5 000 for four years. Kuda invests his money in an account that pays simple interest at 8,3% per annum. At the end of four years, he will receive a bonus of exactly 4% of the accumulated amount. Thabo invests his money in an account that pays interest at 8,1% p.a., compounded monthly.

Whose investment will yield a better return at the end of four years? Justify your answer with appropriate calculations. (5)

6.2 Nine years ago, a bank granted Mandy a home loan of R525 000. This loan was to be repaid over 20 years at an interest rate of 10% p.a., compounded monthly. Mandy's monthly repayments commenced exactly one month after the loan was granted.

6.2.1 Mandy decided to make monthly repayments of R6 000 instead of the required R5 066,36. How many payments will she make to settle the loan? (5)

6.2.2 After making monthly repayments of R6 000 for nine years, Mandy required money to fund her daughter's university fees. She approached the bank for another loan. Instead, the bank advised Mandy that the extra amount repaid every month could be regarded as an investment and that she could withdraw this full amount to fund her daughter's studies. Calculate the maximum amount that Mandy may withdraw from the loan account. (4)

[14]

Exercise 2

QUESTION 1

1.1 Determine the nominal interest rate if the investment received r% p.a. compounded monthly whereas effectively it receives 8,3% per annum. (3)

1.2 Mpho takes a loan of 400 000 at an interest rate of 11% p.a. compounded monthly. Mpho must amortise (pay off) the loan within 5 years with equal monthly repayments starting in one month's time. If he pays the loan over 5 years, his calculations gave him the monthly payments amounting to R8696,97.

1.2.1 Determine the amount of interest Mpho would pay if he were to sign this agreement. (1)

1.2.2 How many full monthly repayments would Mpho pay if he were to increase the monthly payments by 303,03? (4)

1.2.3 What is the value of Mpho's final payment? (4)

1.2.4 How much interest will he save based on the decision he took in 1.2.2? (2)

[14]

QUESTION 2

1.1 Samuel invested an amount with ABC bank at an interest 12% p.a. compounded monthly. His investment grew to R8450 at the end of 10 years. Determine the amount that Samuel initially invested. (3)

1.2 If the inflation rate remains at a constant 4,7 % p.a., what period of time will it take for a certain amount to be worth half of the original amount. (3)

1.3 Lebogo buys a tractor for Rx. She plans to replace this tractor after 5 years. The tractor depreciates by 20% p.a. according to the reducing balance method. The price of a new tractor is expected to increase by 18% p.a. She calculates that if she deposits R8 000 into a sinking fund at the end of each month, it would exactly provide for the shortfall 5 years from now when she has to pay for the new tractor. The bank offers 10% p.a. interest compounded monthly.

1.3.1 Calculate the scrap value of the tractor after 5 years, in terms of x? (1)

1.3.2 Determine the price of the new tractor after 5 years, in terms of x? (1)

1.3.3 Calculate the amount accumulated in the sinking fund after 5 years. (4)

1.3.4 Determine the value of x, the price of the original tractor. . (4)

[16]

QUESTION 3

Jake takes out a bank loan of R600 000 to pay for his new car. He repays the loan with monthly instalments of R9 000, starting one month after the granting of the loan. The interest rate is 13% per annum, compounded quarterly.

3.1 Show that the effective interest rate is 12,86% p.a. compounded monthly. (3)

3.2 How many instalments of R9 000 must be paid? (5)

3.3 What will the final payment be? (5)

3.4 What did the car cost Jake in total by the time it is paid off? (2)

[15]

QUESTION 4

4.1 Hein invests R12 500 for k years at a compound interest rate of 9% p.a. compounded quarterly. At the end of the k years his investment is worth R30 440. Calculate the value of k. (4)

4.2 Matt bought a car for R500 000 on an agreement in which he will repay it in monthly instalments at the end of each month for 5 years. Interest is charged at 18% p.a. compounded monthly.

4.2.1 Calculate the annual effective interest rate of the loan. (3)

4.2.2 Calculate Matt's monthly instalments. (4)

4.2.3 Matt decided to pay R12 700 each month as his repayment. Calculate the outstanding balance of the loan after 2 years. (4)

4.2.4 At the end of the 2 years, the market value of Matt's car had reduced to R304 200. Determine the annual interest rate of depreciation on the reducing value. (2)

[17]

QUESTION 5

A business installs a server for R500 000. The value of the server depreciates at 20% per annum according to the diminishing-balance method.

5.1 Calculate the scrap value of the server at the end of 6 years. (2)

5.2 The server needs to be replaced after 6 years. Calculate the cost of the new server if the inflation rate is at 7% per annum. The older server will be traded in. (3)

5.3 On the day the server gets installed, the business sets up a sinking fund into which equal monthly installments must be paid. Interest on this fund is 8% per annum compounded monthly. The first payment will be made immediately and the last payment will be made at the end of the 6 year period. Calculate the value of the monthly instalment into the sinking fund. (4)

5.4 The business decides to rather pay a monthly instalment of R15 000 into the sinking fund. After how many months will there be more than R1 000 000 in the fund? (5)

[14]

SECTION 5
DIFFERENTIAL CALCULUS

1. Determining the derivative from first principles.

Consider a function $f(x)$, its derivative $f'(x)$ from first is determined using the formula below.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Steps:

- Substitute for both $f(x)$ and $f(x+h)$ in the formula.
- Simplify the numerator by doing the necessary algebraic operations.
- Factor out h in the numerator (where necessary) so that you can cancel with the h in the denominator.
- Substitute for $h = 0$. NOTE: drop the $\boldsymbol{lim}$ when substituting for h.
- Write the final answer.

Examples:

1. $f(x) = 2x + 3$**(linear function)**
   Solution:

$$f(x+h) = 2(x+h) + 3$$
$$f(x+h) = 2x + 2h + 3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(2x + 2h + 3\,) - (2x + 3)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2h}{h}$$

$$f'(x) = \lim\nolimits_{h \to 0} 2$$

$$\therefore f'(x) = 2$$

NOTE:

Limit of a constant is equal to that constant

$$\lim_{x \to 3} a = a$$

2. $f(x) = 3x - x^2$ **(quadratic function)**

Solution:

$$f(x+h) = 3(x+h) - (x+h)^2$$
$$f(x+h) = 3x + 3h - (x^2 + 2xh + h^2)$$
$$f(x+h) = 3x + 3h - x^2 - 2xh - h^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(3x + 3h - x^2 - 2xh - h^2) - (3x - x^2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3h - 2xh - h^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(3 - 2x - h)}{h}$$

$$f'(x) = \lim_{h \to 0} 3 - 2x - h$$

$$f'(x) = 3 - 2x - 0$$

$$\therefore f'(x) = 3 - 2x$$

EXERCISE 1

Determine from first principles, the derivatives of the following functions.

1. $f(x) = x^2 + 3x$
2. $f(x) = 2x^3$
3. $f(x) = \frac{1}{x}$
4. $f(x) = \frac{2}{x^2}$

2. Determining the derivative using rules.

Consider a function $f(x) = ax^n$, its derivative $f'(x)$ is determined as below $\boldsymbol{f'(x) = n.ax^{n-1}}$.

$f'(x)$ means that differentiate $f(x)$ with respect to x.

Explanation:
The exponent drops and multiplies the variable(and it's coefficient). But the exponent of the variable(x) reduces by 1. ie $n \rightarrow (n-1)$
Example:

$$f(x) = 3x^5$$

$f'(x) = 5 \cdot 3x^{5-1}$
$f'(x) = 15x^4$

The derivative can be written in the following ways.

$f'(x)$; $\frac{dy}{dx}$; $\frac{d}{dx}$ or D_x ; this is if the variable is x.

If the variable if for example t; then the above notations will be as
$f'(t)$; $\frac{dy}{dt}$; $\frac{d}{dt}$ or D_t .

So it's very important for you to identify the variable in the expression.

NOTE:

i. the derivative of any constant is equal to zero.
Example
If $y = x^{10} - 4x^3 + 7$ then; $\frac{dy}{dx} = 10x^9 - 12x^2$.
$7 \; becomes \; 0 \; on \; differentiating.$

Why does 7 (a constant) **become 0?**

Remember that any number to the power of 0 is equal to 1. So 7 is the same as 7×1, but $1 = x^0$. So $7 = 7x^0$

$$y = x^{10} - 4x^3 + 7x^0$$
$$\frac{dy}{dx} = 10x^9 - 12x^2 + \mathbf{0} \times \boldsymbol{7x^{-1}}$$
$$but \; \mathbf{0} \times \boldsymbol{7x^{-1}} = \mathbf{0}$$

$\therefore \frac{dy}{dx} = 10x^9 - 12x^2$

ii. $\frac{1}{x^n} = x^{-n}$

$$y = \frac{2}{x^3} - \frac{5}{3x^5} + 20x - 3a$$

$$y = 2x^{-3} - \frac{5x^{-5}}{3} + 20x - 3a$$

$$\frac{dy}{dx} = -6x^{-4} + \frac{25x^{-6}}{3} + 20$$

$$\frac{dy}{dx} = -\frac{6}{x^4} + \frac{25}{3x^6} + 20$$

iii. Expressing a surd as an exponent.

$\sqrt[n]{x} = x^{\frac{1}{n}}$; OR $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

$$y = 5 - x + \sqrt{x^5} - 4.\sqrt[5]{x} + \sqrt[3]{x^2}$$

$$y = 5 - x + x^{\frac{5}{2}} - 4x^{\frac{1}{5}} + x^{\frac{2}{3}}$$

$$\frac{dy}{dx} = -1 + \frac{5}{2}x^{\frac{3}{2}} - \frac{4}{5}x^{-\frac{4}{5}} + \frac{2}{3}x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -1 + \frac{5}{2}x^{\frac{3}{2}} - \frac{4}{5x^{\frac{4}{5}}} + \frac{2}{3x^{\frac{1}{3}}}$$

EXERCISE 2

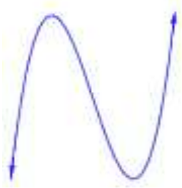
Differentiate the following expressions.

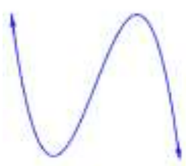
1. $y = 5x + \frac{2}{x^6}$
2. $D_x\left[\sqrt[3]{x} - 3 + 6x^4\right]$
3. $f(x) = ax^2 + bx + c$
4. $\frac{dV}{dr} = \cdots;\ if\ v = \frac{4}{3}\pi r^3 - 2\pi r\,;$

3. Cubic functions and graphs:

$f(x) = ax^3 + bx^2 + cx + d$

If $a > 0$ $\quad$ $a < 0$





The first derivative of $f(x)$ forms a quadratic function

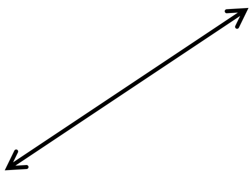
$f'(x) = 3ax^2 + 2bx + c$

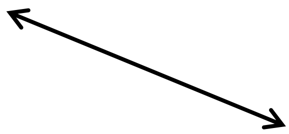
If $a > 0$ $\quad$ $a < 0$

The second derivative of $f(x)$ forms a linear function

$f''(x) = 6ax + b$

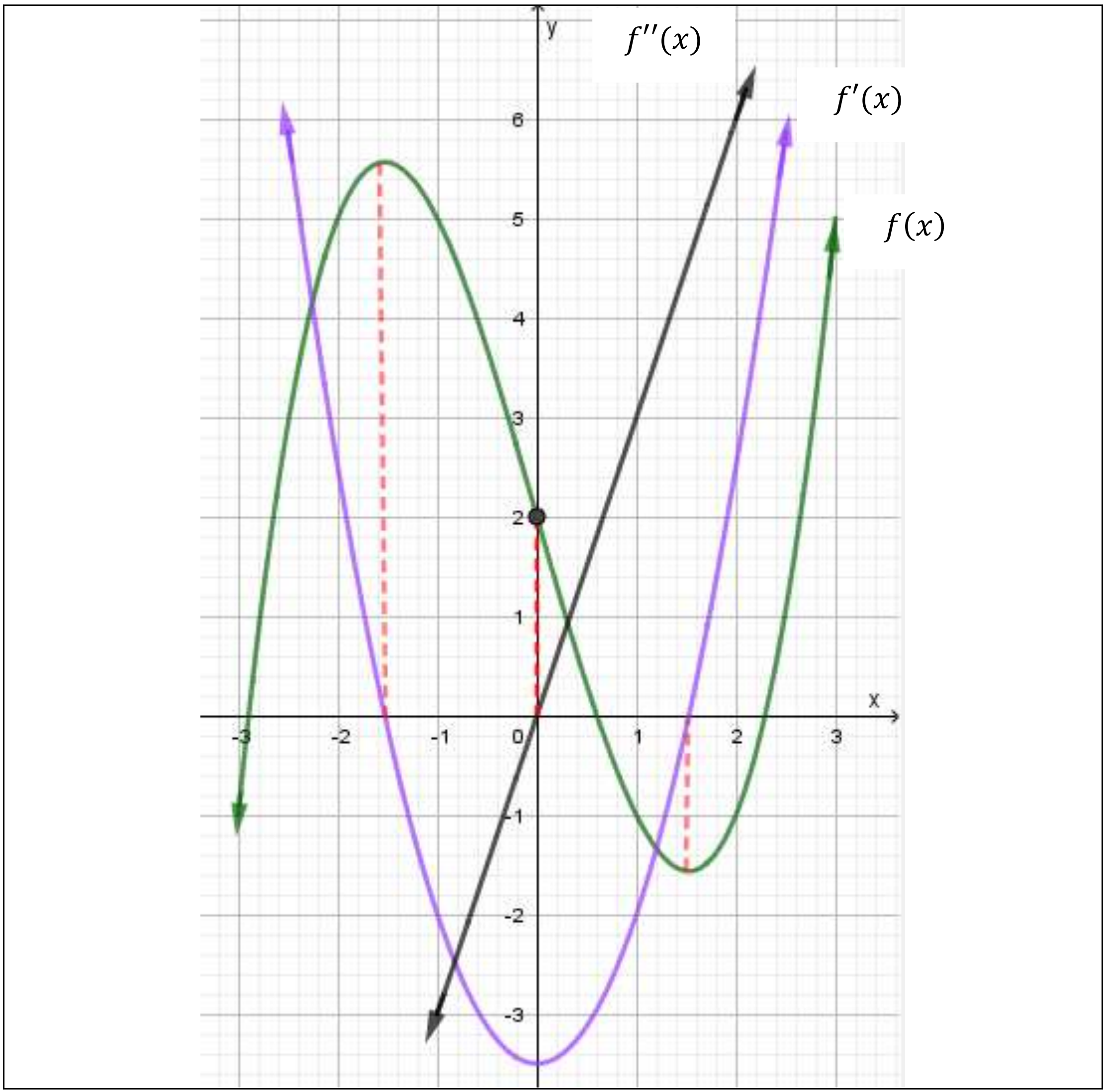
If $a > 0$ $\quad$ $a < 0$





Examples:

1. Given $f(x) = \frac{1}{2}x^3 - \frac{7}{2}x + 2$; on the same set of axes sketch the graph of $f(x)$; $f'(x)$ and $f''(x)$



From the above sketches,

- The x –values at the turning points of $f(x)$ are equal to the x –intercepts of $f'(x)$.
- The x –value at the point of inflection of $f(x)$ is equal to the x –value at the turning point of $f'(x)$
- Also the x –value at the point of inflection of $f(x)$ is equal to the x –intercept $f''(x)$

> The above sketches clearly show the relationship among the three types of functions i.e **Cubic**, **Quadratic/parabolic** and **Linear** function.

The point of Inflection ;Is a point on a cubic graph where the curve changes it's concavity. i.e a point where the graph curves upward if it was originally curving downwards and vice vasa.

For **concave down**, $f''(x) < 0$ and if $f''(x) > 0$ then the graph is **concave up**.

From the above sketch, $f(x)$ is concave up for $x > 0$ and concave down for $x < 0$

Also it can be noted that $f'(x)$ is **decreasing** for $x < 0$ and **increasing** for $x > 0$

2. Sketch: $f(x) = -x^3 + x^2 + 4x - 4$
   y −intercept
   $y = -(0)^3 + (0)^2 + 4(0) - 4$
   $y = -4$
   $(0; -4)$

   x −intercepts
   $-x^3 + x^2 + 4x - 4 = 0$
   $x^3 - x^2 - 4x + 4 = 0$
   $x^2(x-1) - 4(x-1) = 0$
   $(x-1)(x^2-4) = 0$
   $(x-1)(x-2)(x+2) = 0$
   $x = 1\,; x = 2$ or $x = -2$

   $(1;0); (2;0); (-2;0)$

   Turning points

   $f'(x) = -3x^2 + 2x + 4$

But $f'(x) = 0$ at the turning point

$$-3x^2 + 2x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(-3)(4)}}{2(-3)}$$

$x = 1{,}51$ or $x = -0{,}87$

$y = -(1{,}51)^3 + (1{,}51)^2 + 4(1{,}51) - 4 = 0{,}88$

$y = -(-0{,}87)^3 + (-0{,}87)^2 + 4(-0.87) - 4 = -6{,}06$

T.P$(1{,}52; 0{,}88)$ *and* $(-0{,}87; -6{,}06)$

Point of inflection

$f''(x) = -6x + 2$

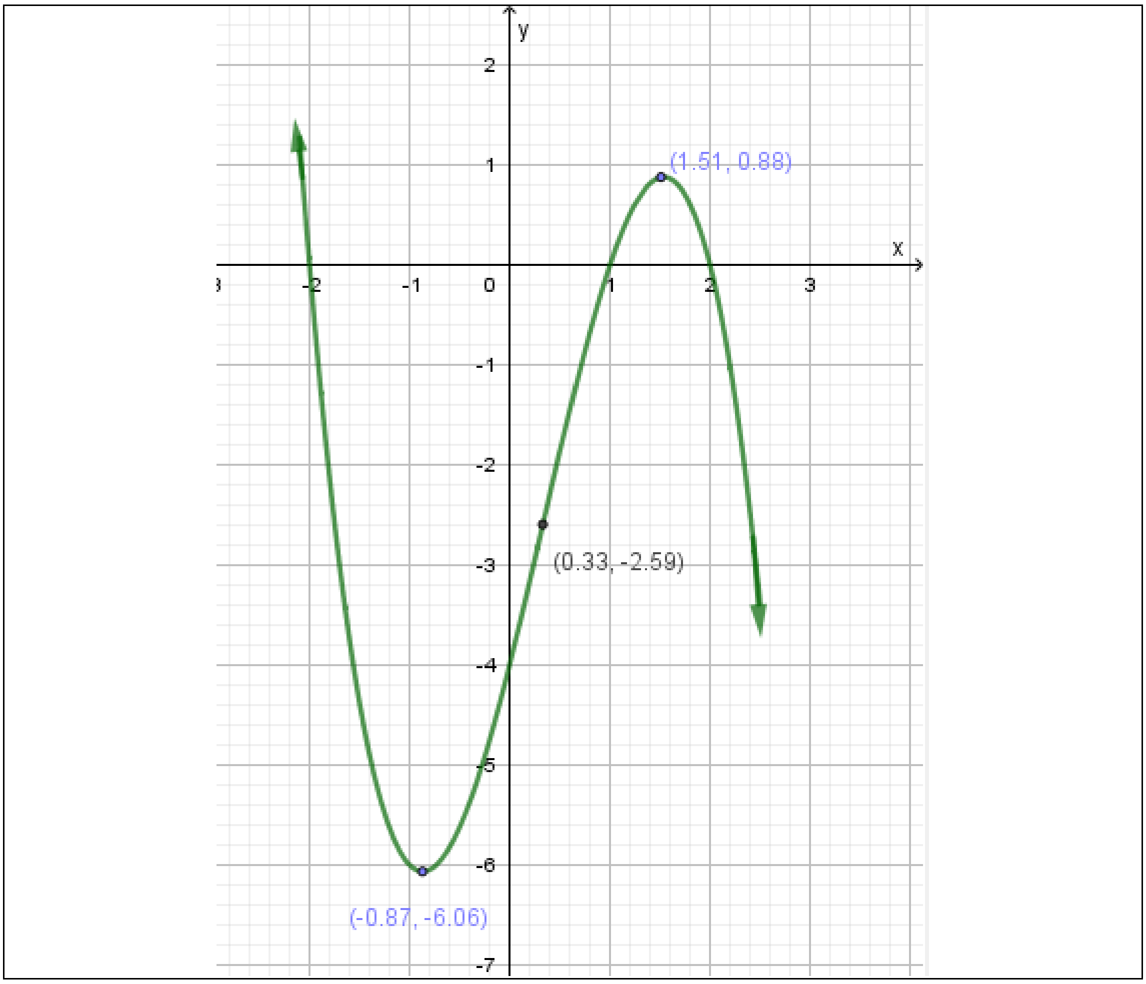
But $f''(x) = 0$ at the point of inflection

$$-6x + 2 = 0$$

$$x = \frac{1}{3}$$

$$y = -\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right) - 4 = -2{,}59$$

$(0{,}33;\ -2{,}59)$



From the above sketch, state the value(s) of x for which

2.1 $f(x) = 0$ (graph on the x −axis)

$x = -2$; $x = 1$ and $x = 2$

2.2 $f(x) > 0$ (graph above the x −axis)

$x < -2$ or $1 < x < 2$

2.3 $f(x) < 0$ (graph below the x −axis)

$-2 < x < 1$ or $x > 2$

2.4 $f(x) \leq 0$ (graph on or below the x −axis)

$-2 \leq x \leq 1$ or $x \geq 2$

2.5 $f(x) \geq 0$ (graph on or above the x −axis)

$x \leq -2$ or $1 \leq x \leq 2$

2.6 $f(x)$ is increasing (when a tangent to the graph has a positive gradient)

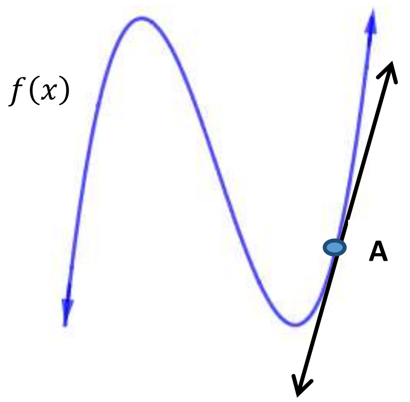
$-0{,}87 < x < 1{,}51$

2.7 $f(x)$ is decreasing (when a tangent to the graph has a negative gradient)
$x < -0{,}87$ or $x > 1{,}51$

2.8 $f(x)$ is concave up (graph curving downwards)
$x < 0{,}33$

2.9 $f(x)$ is concave down (graph curving upwards)
$x > 0{,}33$

4. Tangent to a curve



At the point of tangency **A,** both $f(x)$ and the tangent have equal gradient.

i.e $f'(x) = m_t$

$f'(x)$ the derivative of $f(x)$ represents the gradient at point A

m_t represents the gradient of the tangent at point A

Example

Given: $f(x) = x^2 - 2x + 1$; determine the equation of the tangent to $f(x)$ at a point $x = -1$

Solution

For $x = -1$

$y = (-1)^2 - 2(-1) + 1$

$y = 4$

$(-1; 4)$

$f'(x) = 2x - 2$

$m_t = 2(-1) - 2 = -4$

Using $y = mx + c$ Or $y - y_1 = m(x - x_1)$

$y - y_1 = m(x - x_1)$

$y - 4 = -4(x - -1)$

$y = -4x$

Note: the reason why $f'(x) = 0$ at the turning point of a curve is because at T.P the gradient of both the curve and the tangent is zero.

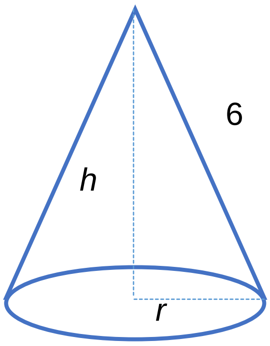
5. **Optimization and the Rates of change.**

Optimizing means to minimize or maximize an application. For example if you are doing a business, you have to minimise the costs but also maximise the profits. So by minimising costs and maximising profits you are trying to optimize your business.
In calculus, we look at the optimization of many real life applications like, costs , profits, volume, surface area, speed, distance etc.

In order to optimize, differentiate the expression, equate it to zero , solve for the variable and then substitute your answer back into the original expression that's if the questions dictates.

Example

A cone has a slant height of 6 units, as shown in the diagram:



The circular base has a radius of *r* and the height of the cone is *h*.

The volume of the cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$

(a) Show that the volume can be written as $V = 12\pi h - \frac{\pi}{3}h^3$

(b) Now calculate the height and radius of the cone that will produce the maximum volume.

(d) Calculate the maximum volume of the cone.

<table>
<tr>
<td>a) $V = \frac{1}{3}\pi r^2 h$
But $r^2 = 36 - h^2$ pythagoras theorem
$V = \frac{1}{3}\pi(36 - h^2)h$
$V = 12\pi h - \frac{\pi}{3}h^3$</td>
<td rowspan="2">c $V_{max} = \frac{1}{3}\pi(2\sqrt{6})^2 . 2\sqrt{3}$
$\therefore V_{max} = \frac{48\pi\sqrt{3}}{3}$</td>
</tr>
<tr>
<td>b) $V = 12\pi h - \frac{\pi}{3}h^3$
$V' = 12\pi - \pi h^2$
$12\pi - \pi h^2 = 0$
$h^2 = 12$
$h = 2\sqrt{3}$ units
Radius
$r = \sqrt{36 - (2\sqrt{3})^2}$
$r = 2\sqrt{6}$ units</td>
</tr>
</table>

Misconceptions and common errors

1.	$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ Some candidates make errors when substituting for $f(x+h)$. E.g for $f(x)=2x^2+3x$ The following errors are likely to be made $f(x+h)=2(x+h)^2+3x$; $f(x+h)=2(x^2+h)+3(x+h)$
2.	Notation errors by some candidates $f(x)=\lim_{h\to 0}=\dfrac{f(x+h)-f(x)}{h}$ Some candidates fail to correctly interpret the minus sign when substituting in the formula . eg $\lim_{h\to 0}\dfrac{2(x+h)^2+3(x+h)-2x^2+3x}{h}$
3.	Some candidates fail to identify a constant term in an expression being differentiated. E.g if $y=3x^2-2x+10a$; then $\frac{dy}{dx}=6x-2+10$
4.	Most candidates think that differentiation is only with respect to x, once the question requires differentiating with respect to another variable then some candidates will fail. E.g if $y=2x^2+5a^4-20$; then $\frac{dy}{da}=\cdots$ in such a case some candidates will not be able to realize that $2x^2$ and -20 are constants and their derivative is 0.
5.	Some candidates write D_x in their final answer after differentiating. This shows that they don't understand what D_x really means. E.g $D_x[-2x^5+3x^2+8]$ $D_x[-10x^4+6x]$wrong because this means that the answer should be differentiated with respect to x Correction $D_x[-2x^5+3x^2+8]$ $-10x^4+6x$

Tips:

- Emphasis should be placed on the use of the correct notation when determining the derivative, either when using first principles or the rules.
- Use brackets when determining the derivative from first principles. This prevents the incorrect simplification that follows.
- Remember that the derivative for a certain value of *x* is the gradient of the tangent to the curve at that point.
- A candidate should know how to determine the turning point and point of inflection. Learners need to be aware of how the first and/or second derivatives change at the turning point and point of inflection.
- It's important that a candidate understands the relationship between the critical points of the graphs of $f(x), f'(x)$ and $f''(x)$
- It's important that candidates do more practice of questions involving the application of calculus in real life and how to apply the condition of optimization.

EXAMINATION QUESTIONS ON DIFFERENTIAL CALCULUS

Nov 2018

QUESTION 8

8.1 Determine $f^{/}(x)$ from first principles if it is given $f(x)=x^2-5$. (5)

8.2 Determine $\frac{dy}{dx}$ if:

8.2.1 $y=3x^3+6x^2+x-4$ (3)

8.2.2 $yx-y=2x^2-2x \quad ; \quad x\neq 1$ (4)

[12]

Nov 2019

QUESTION 7

7.1 Determine $f^{/}(x)$ from first principles if it is given that $f(x)=4-7x$. (4)

7.2 Determine $\frac{dy}{dx}$ if $y=4x^8+\sqrt{x^3}$ (3)

7.3 Given: $y=ax^2+a$

Determine:

7.3.1 $\frac{dy}{dx}$ (1)

7.3.2 $\frac{dy}{da}$ (2)

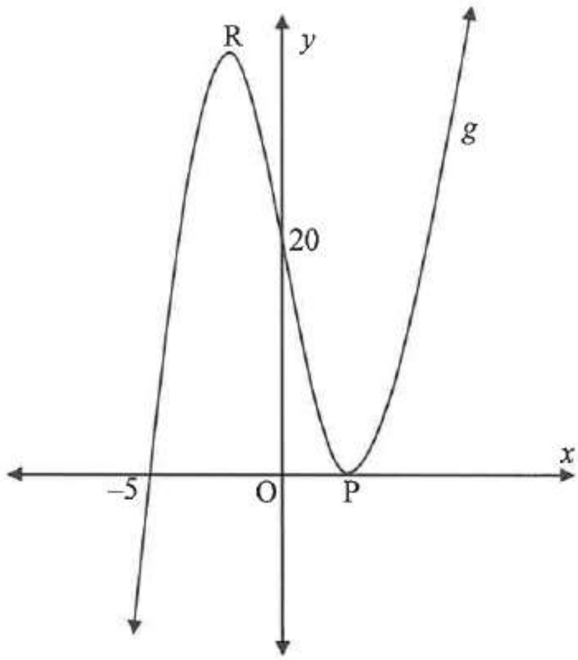
7.4 The curve with equation $y=x+\frac{12}{x}$ passes through the point $A(2\ ;\ b)$. Determine the equation of the line perpendicular to the tangent to the curve at A. (4)

[14]

Nov 2018

QUESTION 9

9.1 The graph of $g(x) = x^3 + bx^2 + cx + d$ is sketched below.
The graph of g intersects the x-axis at $(-5\ ;\ 0)$ and at P, and the y-axis at $(0\ ;\ 20)$.
P and R are turning points of g.



9.1.1 Show that $b = 1$, $c = -16$ and $d = 20$. (4)

9.1.2 Calculate the coordinates of P and R. (5)

9.1.3 Is the graph concave up or concave down at $(0\ ;\ 20)$? Show ALL your calculations. (3)

9.2 If g is a cubic function with:

- $g(3) = g'(3) = 0$
- $g(0) = 27$
- $g''(x) > 0$ when $x < 3$ and $g''(x) < 0$ when $x > 3$,

draw a sketch graph of g indicating ALL relevant points. (3)

[15]

Nov 2019

QUESTION 8

After flying a short distance, an insect came to rest on a wall. Thereafter the insect started crawling on the wall. The path that the insect crawled can be described by $h(t)=(t-6)(-2t^2+3t-6)$, where h is the height (in cm) above the floor and t is the time (in minutes) since the insect started crawling.

8.1 At what height above the floor did the insect start to crawl? (1)

8.2 How many times did the insect reach the floor? (3)

8.3 Determine the maximum height that the insect reached above the floor. (4)
[8]

Nov 2019

QUESTION 9

Given: $f(x)=3x^3$

9.1 Solve $f(x)=f'(x)$ (3)

9.2 The graphs f, f' and f'' all pass through the point $(0\,;0)$.

9.2.1 For which of the graphs will $(0\,;0)$ be a stationary point? (1)

9.2.2 Explain the difference, if any, in the stationary points referred to in QUESTION 9.2.1. (2)

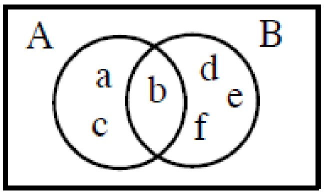
9.3 Determine the vertical distance between the graphs of f' and f'' at $x=1$. (3)

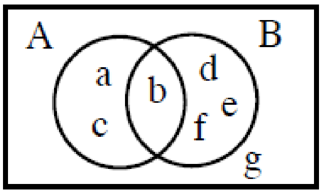
9.4 For which value(s) of x is $f(x)-f'(x)<0$? (4)
[13]

SECTION 6
PROBABILITY & COUNTING PRINCIPLES

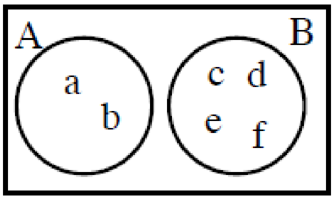
SUMMARY OF ALL PROBABILITY THEORY (GRADE 10 AND 11)

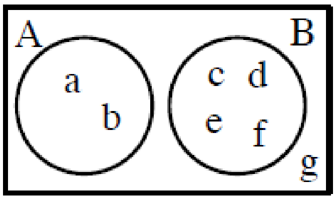
1. Consider the probability rule $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
   A and B are **inclusive** if $P(A \text{ and } B) \neq 0$ (there is an intersection)





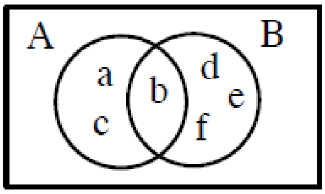
A and B are **mutually exclusive** if $P(A \text{ and } B) = 0$ (there is no intersection)

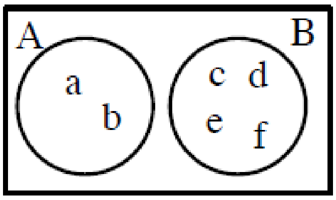




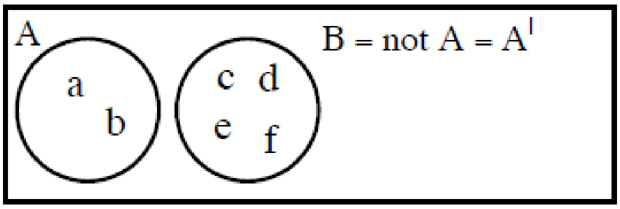
For mutually exclusive events $P(A \text{ or } B) = P(A) + P(B)$

2. **Exhaustive events** contain all elements of the sample space between them.
   In this case, $P(A \text{ or } B) = 1$



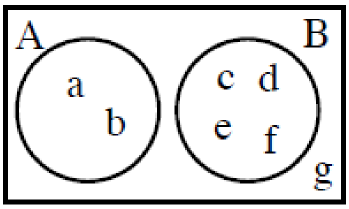


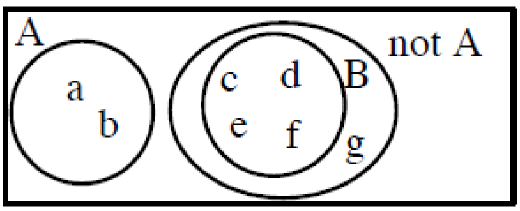
3. **Complementary events** are mutually exclusive and exhaustive.
   In this case, $P(A) + P(B) = 1$



The set {not A} is B since all elements that are not in A are in B.
We can write not $A = B = A^{I}$

4. Where events A and B are not complementary, then the set {not A} will be different to B. $B = \{c\ ;\ d\ ;\ e\ ;\ f\}$ and not $A = \{c\ ;\ d\ ;\ e\ ;\ f\ ;\ g\}$





5. For **independent events**, $P(A \text{ and } B) = P(A) \times P(B)$

Misconceptions and common errors

1.	Some candidates confused mutually exclusive events with independent events.
2.	Some candidates write probability values greater than 1. This shows that they don't know the probability of an event can't be more than 1 or less than 0.

Tips:

- Emphasis should be placed on the understanding of the concepts like mutually exclusive events, independent events and complementary events.
- Where possible, a candidate should use venn diagrams, tree diagrams or contingency table to represent the scenario in the question.

REVISION EXERCISE 1

QUESTION 1

Consider the sample space **(S)** of the natural numbers less than 21. **A** is the event of drawing an even number at random. **B** is the event of drawing a prime number and **C** is the event of drawing a multiple of 5.

1.1 Draw a Venn diagram to illustrate events **A**,**B** and **C**. (6)

1.2 Calculate the following.

1.2.1 P(**A**) (1)

1.2.2 P'(**B**) (2)

1.2.3 P(**A** and **B**) (1)

1.2.4 P(**A** or **B**) (2)

1.3 Are the events **A**, **B** and **C** complementary? Give a reason for your answer. (2)

[16]

QUESTION 2

2.1 At a certain school there are 64 boys in Grade 10. Their sport preferences are indicated below:

- 24 boys play soccer
- 28 boys play rugby
- 10 boys play both soccer and rugby
- 22 boys do not play soccer or rugby

2.1.1 Represent the information above in a Venn diagram. (3)

2.1.2 Calculate the probability that a Grade 10 boy at the school, selected at random, plays:

(a) Soccer and rugby (3)

(b) Soccer or rugby (1)

2.1.3 Are the events a Grade 10 boy plays soccer at the school and a Grade 10 boy plays rugby at the school, mutually exclusive? Justify your answer. (2)

2.2 One morning Aslam conducted a survey in his residential area to establish how many passengers, excluding the drivers, travel in a car. The results are shown in the table below:

Number of passengers, excluding the driver	0	1	2	3	4
Number of cars	7	11	6	5	1

Calculate the probability that, excluding the driver, there are more than two passengers in a car. (3)

2.3 If you throw two dice at the same time, the probability that a six will be shown on one of the two dice is $\frac{10}{36}$ and the probability that a six be shown on both the dices is $\frac{1}{36}$.

What is the probability that a six will NOT show on either of the dice when you throw two dice at the same time? (3)

[15]

REVISION EXERCISE 2

QUESTION 1

Matthew has three R100 notes, five R50 notes and seven R20 notes in his drawer. He is in a hurry to watch a film and grabs two notes out of his drawer.

1.1 Draw a tree diagram of the situation. (3)

1.2 What is the probability that:

1.2.1 both notes are R100 notes? (1)

1.2.2 one note is a R100 note and the other is a R20? (2)

1.2.3 the total amount is more than R100? (2)

[8]

QUESTION 2

A travel agent did a survey amongst his clients as to which type of holiday they prefer.

	Game reserve	Sea	Travel	Total
Male	450	100	a	700
Female	150	150	75	b
Total	600	c	d	e

Event A: a person is male

Event B: a person prefers a game reserve holiday

2.1 Determine the values of a, b, c, d and e in the two-way contingency table. (5)

2.2 Are events A and B mutually exclusive? Explain your answer. (2)

2.3 Are events A and B independent? Show the necessary calculations. (4)

2.4 If a person is selected at random, what is the probability that the person:

2.4.1 is not male and prefers travelling holidays. (1)

2.4.2 prefers game reserve or sea holidays? (2)

2.4.3 is female or prefers sea holidays: (2)

[16]

EXAMINATION QUESTIONS ON PROBABILITY

Nov 2019

QUESTION 10

The school library is open from Monday to Thursday. Anna and Ben both studied in the school library one day this week. If the chance of studying any day in the week is equally likely, calculate the probability that Anna and Ben studied on:

10.1 The same day (2)

10.2 Consecutive days (3)
[5]

QUESTION 11

11.1 Events **A** and **B** are independent. $P(A) = 0{,}4$ and $P(B) = 0{,}25$.

11.1.1 Represent the given information on a Venn diagram. Indicate on the Venn diagram the probabilities associated with each region. (3)

11.1.2 Determine $P(\mathbf{A} \text{ or NOT } \mathbf{B})$. (2)

11.2 Motors Incorporated manufacture cars with 5 different body styles, 4 different interior colours and 6 different exterior colours, as indicated in the table below.

BODY STYLES	INTERIOR COLOURS	EXTERIOR COLOURS
Five body styles	Blue	Silver
	Grey	Blue
		White
	Black	Green
	Red	Red
		Gold

The interior colour of the car must NOT be the same as the exterior colour.

Motors Incorporated wants to display one of each possible variation of its car in their showroom. The showroom has a floor space of $500\ m^2$ and each car requires a floor space of $5\,m^2$.

The interior colour of the car must NOT be the same as the exterior colour.

Motors Incorporated wants to display one of each possible variation of its car in their showroom. The showroom has a floor space of $500\ m^2$ and each car requires a floor space of $5\,m^2$.

Is this display possible? Justify your answer with the necessary calculations. (6)
[11]

Nov 2018

QUESTION 12

12.1 Given: $P(A) = 0,45$; $P(B) = y$ and $P(A \text{ or } B) = 0,74$

Determine the value(s) of y if A and B are mutually exclusive. (3)

12.2 An organisation decided to distribute gift bags of sweets to a Grade R class at a certain school. There is a mystery gift in exactly $\frac{1}{4}$ of the total number of bags.

Each learner in the class may randomly select two gift bags of sweets, one after the other. The probability that a learner selects two bags of sweets with a mystery gift is $\frac{7}{118}$. Calculate the number of gift bags of sweets with a mystery gift inside. (6)

[9]

COUNTING PRINCIPLES

RULE 1

If one operation can be done in m ways and a second operation can be done in n ways then the total possible number of different ways in which both operations can be done is $m \times n$.

RULE 2

The number of arrangements of n different things taken in n ways is: $n!$ (n factorial)

Example:

In how many ways can 6 different people be seated in the first six seats in a movie theatre?

$$\mathbf{6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720}$$

RULE 3

The number of arrangements of n different things taken r at a time is given by

$$\frac{n!}{(n-r)!}$$

Example:

In how many ways can 7 vacant places be filled by 10 different people?

$$\frac{10!}{(10-7)!} = \frac{10!}{3!} = 604\ 800$$

RULE 4

The number of different ways that n letters can be arranged where m_1 of the letters are identical, m_2 of the letters are identical, m_3 of the letters are identical, …….,m_n of the letters are identical is given by:

$$\frac{n!}{m_1! \times m_2! \times m_3! \times \ldots \ldots . \times m_n!}$$

where the repeated letters are treated as identical.

Example:

Consider the letters of the word **NEEDED.**

How many word arrangements can be made with this word if the repeated letters are treated as identical?

$$\frac{6!}{3! \times 2!} = 60$$

More examples:

1. Consider the word **LOVERS**.
   a. How many six-letter word arrangements can be made if the letters may be repeated?
   b. How many six-letter word arrangements can be made if the letters may not be repeated?
   c. How many four-letter word arrangements can be made if the letters may be repeated?
   d. How many four-letter word arrangements can be made if the letters may not be repeated?

Solutions:

(a) When the letters may be repeated, we use **exponential** notation:

$$6 \times 6 \times 6 \times 6 \times 6 \times 6 \ = \ 6^6 \ = \ 46\,656$$

(b) When letters may not be repeated, we use **factorial** notation:

$$6 \times 5 \times 4 \times 3 \times \ 2 \times 1 \ = \ 6! = \ 720$$

(c) $6 \times 6 \times 6 \times 6 \ = \ 6^4 \ = 1\,296$

(d) $\frac{6!}{(6-4)!} = 360$

2. Three Mathematics books and five Science books are to be arranged on a shelf.

(a) In how many ways can these books be arranged if they are treated as separate books?

(b) In how many ways can these books be arranged if they are treated as identical books?

Solutions:

(a) $8! = 40\ 320$

(b) $\frac{8!}{3!\times 5!} = 56$

EXAMINATION QUESTIONS ON COUNTING PRINCIPLES

Nov 2018

QUESTION 11

Given the digits: 3 ; 4 ; 5 ; 6 ; 7 ; 8 and 9

11.1 Calculate how many unique 5-digit codes can be formed using the digits above, if:

11.1.1 The digits may be repeated (2)

11.1.2 The digits may not be repeated (2)

11.2 How many unique 3-digit codes can be formed using the above digits, if:

- Digits may be repeated
- The code is greater than 400 but less than 600
- The code is divisible by 5 (3)

[7]

Nov 2015

11.1 For two events, A and B, it is given that:

$P(A) = 0,2$
$P(B) = 0,63$
$P(A \text{ and } B) = 0,126$

Are the events, A and B, independent? Justify your answer with appropriate calculations. (3)

11.2 The letters of the word DECIMAL are randomly arranged into a new 'word', also consisting of seven letters. How many different arrangements are possible if:

11.2.1 Letters may be repeated (2)

11.2.2 Letters may not be repeated (2)

11.2.3 The arrangements must start with a vowel and end in a consonant and no repetition of letters is allowed (4)

11.3 There are t orange balls and 2 yellow balls in a bag. Craig randomly selects one ball from the bag, records his choice and returns the ball to the bag. He then randomly selects a second ball from the bag, records his choice and returns it to bag. It is known that the probability that Craig will select two balls of the same colour from the bag is 52%.

Calculate how many orange balls are in the bag. (6)
[17]

Prelim Gauteng 2016

QUESTION 12

Given the word: E D U C A T I O N

12.1 In how many unique ways can all the letters in the word above be arranged? (1)

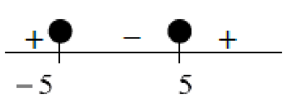
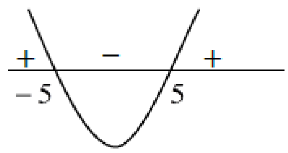
12.2 If 5 letters are randomly chosen from the word "EDUCATION", determine how many unique 5-letter arrangements can be formulated? (3)
[4]

SOLUTIONS

SECTION1

ALGEBRA, EQUATIONS & INEQUALITIES

Feb- March 2018

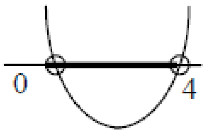
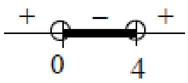
1.1.1	$x^2 - 6x - 16 = 0$ $(x-8)(x+2) = 0$ $x = -2$ or $x = 8$	✓ factors ✓ $x = -2$ ✓ $x = 8$ (3)
1.1.2	$2x^2 + 7x - 1 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-(7) \pm \sqrt{(7)^2 - 4(2)(-1)}}{2(2)}$ $= \frac{-7 \pm \sqrt{57}}{4}$ $x = 0{,}14$ or $x = -3{,}64$	✓subs into correct formula ✓ $\frac{-7 \pm \sqrt{57}}{4}$ ✓ $x = 0{,}14$ ✓ $x = -3{,}64$
	OR/*OF* NOTE: Penalise 1 mark if the rounding to TWO decimal places is incorrect. $x^2 + \frac{7}{2}x + \frac{49}{16} = \frac{1}{2} + \frac{49}{16}$ $\left(x + \frac{7}{4}\right)^2 = \frac{57}{16}$ $x + \frac{7}{4} = \pm\frac{\sqrt{57}}{4}$ $x = \frac{-7 \pm \sqrt{57}}{4}$ $x = 0{,}14$ or $x = -3{,}64$ **Practice rounding off to the correct decimal places to avoid losing marks due to incorrect rounding off**	**OR/*OF*** ✓for adding $\frac{49}{16}$ on both sides ✓ $\frac{-7 \pm \sqrt{57}}{4}$ ✓ $x = 0{,}14$ ✓ $x = -3{,}64$ (4)
1.2	$x^2 - 25 < 0$ $(x-5)(x+5) < 0$   $-5 < x < 5$ $x = \{-4 ; -3 ; -2 ; -1 ; 0 ; 1 ; 2 ; 3 ; 4\}$ NOTE: Final answer only 2 / 2	✓factors ✓✓inequality ✓answer (4)

1.3	$x = 2y - 1$ $(2y-1)^2 - 7 - y^2 = -y$ $4y^2 - 4y + 1 - 7 - y^2 = -y$ $3y^2 - 3y - 6 = 0$ $y^2 - y - 2 = 0$ $(y-2)(y+1) = 0$ $y = 2$ or $y = -1$ $x = 2(2) - 1$ or $x = 2(-1) - 1$ $x = 3$ or $x = -3$	✓ $x = 2y - 1$ ✓ substitution ✓ correct standard form ✓ factors ✓ y – values ✓ x – values
	OR/*OF* $y = \frac{x+1}{2}$ $x^2 - 7 - y^2 = -y$ $x^2 - 7 - \left(\frac{x+1}{2}\right)^2 = -\left(\frac{x+1}{2}\right)$ $x^2 - 7 - \left(\frac{x^2+2x+1}{4}\right) = \frac{-x-1}{2}$ $4x^2 - 28 - x^2 - 2x - 1 = -2x - 2$ $3x^2 - 27 = 0$ $x^2 - 9 = 0$ $(x-3)(x+3) = 0$ $x = -3$ or $x = 3$ $y = \frac{-3+1}{2}$ or $y = \frac{3+1}{2}$ $y = -1$ or $y = 2$	**OR/*OF*** ✓ $y = \frac{x+1}{2}$ ✓ substitution ✓ correct standard form ✓ factors ✓ x – values ✓ y – values (6)
1.4	$\frac{3^{2018} + 3^{2016}}{3^{2017}}$ $= \frac{3^{2017}(3^1 + 3^{-1})}{3^{2017}}$ $= 3 + \frac{1}{3}$ $= 3\frac{1}{3}$ or $\frac{10}{3}$ **OR/*OF***	✓ common factor 3^{2017} ✓ answer **OR/*OF***

	$\dfrac{3^{2018}+3^{2016}}{3^{2017}}$ $=\dfrac{3^{2016}(3^2+1)}{3^{2017}}$ $=\dfrac{10}{3}$ **OR/*OF*** $\dfrac{3^{2018}+3^{2016}}{3^{2017}}$ $=\dfrac{3^{2018}}{3^{2017}}+\dfrac{3^{2016}}{3^{2017}}$ $=3+\dfrac{1}{3}$ $=3\dfrac{1}{3}$ or $\dfrac{10}{3}$	✓ common factor 3^{2016} ✓ answer **OR/*OF*** ✓ dividing by 3^{2017} ✓ answer (2)
1.5.1	$3x-5\geq 0$ and $x\neq 3$ $x\geq\dfrac{5}{3}$ and $x\neq 3$	✓ $3x-5\geq 0$ ✓ $x\geq\dfrac{5}{3}$ ✓ $x\neq 3$ (3)
1.5.2	$\dfrac{\sqrt{3x-5}}{x-3}=1$ $\sqrt{3x-5}=x-3$ $3x-5=(x-3)^2$ $3x-5=x^2-6x+9$ $x^2-9x+14=0$ $(x-7)(x-2)=0$ $x\neq 2$ or $x=7$ NOTE: If $x=2$ is not rejected, then maximum 3 / 4 marks	✓ $\sqrt{3x-5}=x-3$ ✓ $3x-5=(x-3)^2$ ✓ factors ✓ $x=7$ (4) **[26]**

When dealing with equations/inequalities involving surds, it's important to check your x values by substituting back into the original equation/inequality to find out if they satisfy the given equation/ inequality

Feb /March 2017 Solution

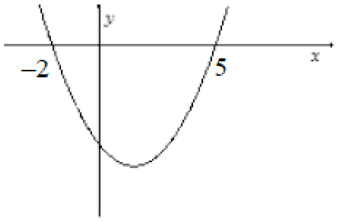
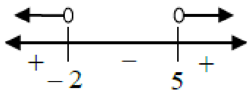
1.1.1	$(x-3)(x+1)=0$ $x=3$ or $x=-1$	✓ answer ✓ answer (2)
1.1.2	$\sqrt{x^3}=512$ $x^{\frac{3}{2}}=512$ $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}=\left(8^3\right)^{\frac{2}{3}}$ $x=64$ **OR** $\sqrt{x^3}=512$ $x^3=262144$ $x^3=2^{18}$ $x=2^6$ $x=64$	✓ $x^{\frac{3}{2}}$ ✓ $\left(8^3\right)^{\frac{2}{3}}$ ✓ answer (3) ✓ squaring both sides ✓ $x^3=2^{18}$ ✓ answer (3)
1.1.3	$x(x-4)<0$  **OR / OF**  $0<x<4$ **OR / OF** $x\in(0;\,4)$	✓critical values ✓inequality or interval (2)

1.2.1	$x^2-5x+2=0$ $x=\dfrac{5\pm\sqrt{(-5)^2-4(1)(2)}}{2(1)}$ $x=\dfrac{5\pm\sqrt{17}}{2}$ $x=0{,}44$ or $x=4{,}56$	✓ subst correct formula ✓ answer ✓ answer (3)
	OR $x^2-5x+2=0$ $x^2-5x=-2$ $x^2-5x+\left(-\dfrac{5}{2}\right)^2=-2+\left(-\dfrac{5}{4}\right)^2$ $\left(x-\dfrac{5}{2}\right)^2=\dfrac{17}{4}$ $x=\dfrac{5+\sqrt{17}}{2}$ or $x=\dfrac{5-\sqrt{17}}{2}$ $x=0{,}44$ or $x=4{,}56$ Completing squares	✓ $\left(x-\dfrac{5}{2}\right)^2=\dfrac{17}{4}$ ✓ answer ✓ answer (3)
1.2.2	$f(x)=x^2-5x+2$ $x^2-5x+2=c$ $x^2-5x+2-c=0$ $b^2-4ac<0$ $(-5)^2-4(1)(2-c)<0$ $25-8+4c<0$ $4c<-17$ $c<-\dfrac{17}{4}$	✓ standard form ✓ $b^2-4ac<0$ ✓ substitution ✓ answer (4)
1.3	$x=2y+2$ $x^2-2xy+3y^2=4$ $(2y+2)^2-2y(2y+2)+3y^2=4$ $4y^2+8y+4-4y^2-4y+3y^2=4$ $3y^2+4y=0$ $y(3y+4)=0$ $y=0$ or $y=-\dfrac{4}{3}$ $x=2$ $\quad x=-\dfrac{2}{3}$	✓ substitution ✓ simplification ✓ standard form ✓ factors ✓ $y=0$; $y=-\dfrac{4}{3}$ ✓ x-values (ca on both x-values) (6)

1.4	$S = \dfrac{6}{x^2+2}$ For S to be a maximum the denominator needs to be at a minimum. *Vir S om 'n maksimum waarde te hê, moet die deler 'n minimum waarde h* Minimum of x^2+2 is 2 $\text{Maximum of } S = \dfrac{6}{x^2+2}$ $= \dfrac{6}{2}$ $= 3$	✓ Minimum of x^2+2 is 2 ✓3 (2)
		[22]

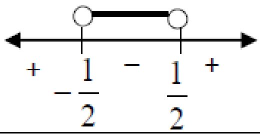
Nov 2018 Solutions

1.1.1	$x^2-4x+3=0$ $(x-3)(x-1)=0$ $x=3$ or $x=1$	✓factors/correct subt in formula ✓ $x=3$ ✓ $x=1$ (3)
1.1.2	$5x^2-5x+1=0$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\dfrac{5\pm\sqrt{25-4(5)(1)}}{2(5)}$ $=\dfrac{5\pm\sqrt{5}}{10}$ $x=0{,}72$ or $x=0{,}28$	✓substitution into the correct formula ✓ $x=0{,}72$ ✓ $x=0{,}28$ (3)

1.1.3	$x^2 - 3x - 10 > 0$ $(x-5)(x+2) > 0$ **OR/OF**   $x < -2$ or $x > 5$	✓factors/ critical values ✓✓ $x < -2$ or $x > 5$ (3)
1.1.4	$3\sqrt{x} = x - 4$ $9x = x^2 - 8x + 16$ $x^2 - 17x + 16 = 0$ $(x-16)(x-1) = 0$ $x = 16$ or $x = 1$ NA	✓ squaring both sides ✓ $x^2 - 17x + 16 = 0$ ✓ factors ✓ answer with selection (4)
	OR/OF $3x^{\frac{1}{2}} = x - 4$ $x - 3x^{\frac{1}{2}} - 4 = 0$ $\left(x^{\frac{1}{2}} - 4\right)\left(x^{\frac{1}{2}} + 1\right) = 0$ $x^{\frac{1}{2}} = 4$ or $x^{\frac{1}{2}} = -1$ $x = 16$ NA	**OR/OF** ✓ standard form ✓ recognize $x = \left(x^{\frac{1}{2}}\right)^2$ ✓ factors ✓ answer with selection (4)
1.2	$2y + 9x^2 = -1$......(1) $3x - y = 2$ ...... (2) $y = 3x - 2$ ....(3) $2(3x-2) + 9x^2 = -1$ $6x - 4 + 9x^2 = -1$ $9x^2 + 6x - 3 = 0$ $3x^2 + 2x - 1 = 0$ $(3x-1)(x+1) = 0$ $x = \frac{1}{3}$ or $x = -1$ $y = -1$ or $y = -5$	✓ $y = 3x - 2$ ✓substitution ✓ standard form ✓factors ✓both x values ✓both y values

1.3	$3^{9x}=64$ $\left(3^{3x}\right)^3=(4)^3$ $3^{3x}=4$ $5^{\sqrt{p}}=64$ $\sqrt{5}^{\sqrt{p}}=\sqrt{64}$ $\sqrt{5}^{\sqrt{p}}=8$ $\frac{\left[3^{x-1}\right]^3}{\sqrt{5}^{\sqrt{p}}}=\frac{3^{3x-3}}{\sqrt{5}^{\sqrt{p}}}$ **OR/OF** $=\frac{3^{3x}.3^{-3}}{5^{\frac{\sqrt{p}}{2}}}$ $=\frac{3^{3x}}{27\times\sqrt{5}^{\sqrt{p}}}$ $=\frac{\sqrt[3]{64}.3^{-3}}{\sqrt{64}}$ $=\frac{4}{27\times 8}$ $=\frac{1}{54}$	✓ $3^{3x}=4$ ✓ $\sqrt{5}^{\sqrt{p}}=8$ ✓ 3^{3x-3} or $3^{3x}.3^{-3}$ ✓answer (4)
	OR/OF $\frac{\left(3^{x-1}\right)^3}{\sqrt{5}^{\sqrt{p}}}=\frac{3^{3x}.3^{-3}}{\left(5^{0.5}\right)^{\sqrt{p}}}$ $=\frac{3^{3x}.3^{-3}}{\left(5^{\sqrt{p}}\right)^{0.5}}$ $=\frac{4.3^{-3}}{\sqrt{64}}$ $=\frac{4.\frac{1}{27}}{8}=\frac{1}{54}$	**OR/OF** ✓ 3^{3x-3} or $3^{3x}.3^{-3}$ ✓ $3^{3x}=4$ ✓ $\sqrt{5}^{\sqrt{p}}=8$ ✓answer (4)
		[23]

Nov 2019 Solutions

1.1.1	$x^2+5x-6=0$ $(x+6)(x-1)=0$ $x=-6$ or $x=1$	✓factors ✓ $x=-6$ ✓ $x=1$ (3)
1.1.2	$4x^2+3x-5=0$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\dfrac{-3\pm\sqrt{(3)^2-4(4)(-5)}}{2(4)}$ $x=\dfrac{-3\pm\sqrt{89}}{8}$ $x=-1{,}55$ or $x=0{,}8$	✓substitution into the correct formula ✓ $x=-1{,}55$ ✓ $x=0{,}8$ (3)
1.1.3	$4x^2-1<0$ $(2x+1)(2x-1)<0$ $\dfrac{-1}{2}<x<\dfrac{1}{2}$  	✓factors ✓method ✓answer (3)
1.1.4	$\left(\sqrt{\sqrt{32}+x}\right)\left(\sqrt{\sqrt{32}-x}\right)=x$ $\sqrt{32-x^2}=x$ $32-x^2=x^2$ $-2x^2=-32$ $x^2=16$ $x=\pm4$ $\therefore x=4$	✓ $\sqrt{32-x^2}$ ✓ squaring both sides ✓ $x^2=16$ ✓ $x=4$ (selection) (4)
1.2	$y+x=12$ $y=-x+12$..........(1) $xy=14-3x$........(2) Sub (1) into (2) $x(-x+12)=14-3x$ $-x^2+12x-14+3x=0$ $-x^2+15x-14=0$ $x^2-15x+14=0$ $(x-14)(x-1)=0$ $x=14$ or $x=1$ $y=-2$ or $y=11$	✓y subject of the formula ✓ substitution ✓simplification ✓both values of x ✓both values of y (5)

1.3	3 6 9 12 15 18 21 24 27 30 3 3 3^2 3 3 3^2 3 3 3^3 3 $\therefore k = 14$	✓identifying multiples of 3 ✓ten multiples of 3 ✓ powers of 3 ✓ answer (4)
		[22]

More about Qn 1.3

Determine the maximum value of $\boldsymbol{k}$ such that $\boldsymbol{p}^{k}$is a factor of $\boldsymbol{n}$!.

<u>Solution</u>

For $\boldsymbol{k}; \boldsymbol{p}$ and $\boldsymbol{n} \in \mathbb{R}$also $\boldsymbol{p}$ being a prime number.

Using the ladder or factor tree method of factorization , divide $n$by $p$and only record the whole number quotients. Continue dividing the subsequent quotients by $\boldsymbol{p}$until the last value is zero. Sum all the quotients for the value of $\boldsymbol{k}$.

Examples

1. Determine the maximum value of k such that 3^k is a factor of 30!.

3	30
3	**10**
3	**3**
3	**1**
	0

$k = 10 + 3 + 1$

$k = 14$

$\therefore 3^{14}$is a factor of $30!$

2. Determine the maximum value of k such that 2^k is a factor of 30!.

2	30
2	**15**
2	**7**
2	**3**
2	**1**
	0

$k = 15 + 7 + 3 + 1$

$k = 26$

$\therefore 2^{26}$is a factor of $30!$

SECTION 2

SEQUENCES AND SERIES

Feb-March 2018 Solutions

2.1.1	$30\ ;\ 10\ ;\ \frac{10}{3}\ ....$ $a=30 \quad r=\frac{1}{3}$ $T_n=ar^{n-1}$ $\frac{10}{729}=30\left(\frac{1}{3}\right)^{n-1}$ $\frac{1}{2187}=3^{1-n}$ $3^{-7}=3^{1-n}$ $-7=1-n$ $n=8$ **OR/*OF*** $\frac{1}{2187}=\left(\frac{1}{3}\right)^{n-1}$ $\left(\frac{1}{3}\right)^{7}=\left(\frac{1}{3}\right)^{n-1}$ $7=n-1$ $n=8$	✓ $r=\frac{1}{3}$ ✓substitution into correct formula ✓ $3^{-7}=3^{1-n}$ or $\left(\frac{1}{3}\right)^{7}=\left(\frac{1}{3}\right)^{n-1}$ or use of logs ✓ $n=8$ (4)
2.1.2	$S_\infty=\frac{a}{1-r}$ $=\frac{30}{1-\frac{1}{3}}$ $=45$	✓substitution into correct formula ✓answer (2)
2.2	$S_n=a+(a+d)+......+(a+(n-2)d)+(a+(n-1)d)$ (1) $S_n=(a+(n-1)d)+(a+(n-2)d)+.........+(a+d)+a$ (2) Adding both equations/*Tel die twee vergelykings bymekaar:* $2S_n=2a+(n-1)d+2a+(n-1)d+2a+(n-1)d+.............$ $=n[2a+(n-1)d]$ $S_n=\frac{n}{2}[2a+(n-1)d]$ **OR/OF** $S_n=a+(a+d)+......+(a+(n-2)d)+T_n$ (1) $S_n=T_n+(T_n-d)+(T_n-2d)+.........+a$ (2) Adding both equations/*Tel die twee vergelykings bymekaar:* $2S_n=(a+T_n)+(a+T_n)+(a+T_n)+.....+(a+T_n)$ $S_n=\frac{n}{2}(a+T_n)$ but $Tn=a+(n-1)d$ $S_n=\frac{n}{2}[2a+(n-1)d]$	✓expanding S_n ✓reverse writing ✓ $2S_n=n[2a+(n-1)d]$ ✓ $S_n=\frac{n}{2}[2a+(n-1)d]$ (4) ✓expanding S_n ✓reverse writing ✓ $2S_n=n(a+T_n)$ ✓ $S_n=\frac{n}{2}[2a+(n-1)d]$ (4) **[10]**

Feb – March 2017 Solutions

2.1	For geometric: $-\frac{1}{4}\ ;\ b\ ;-1\ ;\ ......$ $\frac{b}{-\frac{1}{4}}=-\frac{1}{b}$ $b^2=\frac{1}{4}$ $b=\pm\frac{1}{2}$ **OR** $b=\pm\sqrt{\left(-\frac{1}{4}\right)(-1)}$ $b=\pm\frac{1}{2}$	✓ $\frac{b}{-\frac{1}{4}}=-\frac{1}{b}$ ✓ $b=\frac{1}{2}$ ✓ $b=-\frac{1}{2}$ (3) ✓ $b=\pm\sqrt{\left(-\frac{1}{4}\right)(-1)}$ ✓ $b=\frac{1}{2}$ ✓ $b=-\frac{1}{2}$ (3)

2.2	$-\frac{1}{4};\ \frac{1}{2};\ -1;\ \ldots\ldots$ $r=-2$ $T_{19}=ar^{18}$ $=\left(-\frac{1}{4}\right)(-2)^{18}$ $=\left(-\frac{2^{18}}{2^{2}}\right)$ $=-2^{16}$ $=-65536$ **OR / OF** $T_{19}=ar^{18}$ $=\left(-\frac{1}{4}\right)(-2)^{18}$ $=\left(-2^{-2}\right)\left(2^{18}\right)$ $=-2^{16}$ $=-65536$	✓$r=-2$ ✓subst. into correct formula ✓-65536 / -2^{16} (3) ✓$r=-2$ ✓subst. into correct formula ✓-65536 / -2^{16} (3)
2.3	The series is: $-\frac{1}{4};\ \frac{1}{2};\ -1;\ 2;\ -4;\ 8;\ \ldots\ldots$ The new positive term series: $\frac{1}{2};\ 2;\ 8;\ 32;\ 128;\ \ldots\ldots$ $a=\frac{1}{2}$ $\quad r=4$ $\sum_{n=1}^{20}\left(\frac{1}{2}\right)(4)^{n-1}$ **OR/*OF*** $\sum_{p=0}^{19}\left(\frac{1}{2}\right)(4)^{p}$ **etc.**	✓ $a=\frac{1}{2}$ ✓ $r=4$ ✓ $\sum_{n=1}^{20}$ or $\sum_{p=0}^{19}$ ✓ correct formula (4)
2.4	No, the series is not convergent / *Nee, die reeks konvergeer nie* $r=4$ and for convergence $-1<r<1$ $r=4$ *en vir konvergering* $-1<r<1$	✓no ✓reason (2)
		[12]

Nov 2018 Solutions

2.1.1	42	✓answer (1)
2.1.2	$2a=6$ $a=3$ $3a+b=1$ $3(3)+b=1$ $b=-8$ $a+b+c=2$ $(3)+(-8)+c=2$ $c=7$ $T_n=3n^2-8n+7$ **OR/OF** $2a=6$ $a=3$ $T_n=3n^2+bn+c$ $T_1:3+b+c=2$ $\quad b+c=-1$ ......(1) $T_2:12+2b+c=3$ $\quad 2b+c=-9$ ......(2) T_2-T_1: $b=-8$ Subst. in (1): $-8+c=-1$ $c=7$ $T_n=3n^2-8n+7$	✓ $a=3$ ✓ $b=-8$ ✓ $c=7$ ✓ $T_n=an^2+bn+c$ (4) **OR/OF** ✓ $a=3$ ✓ $b=-8$ ✓ $c=7$ ✓ $T_n=an^2+bn+c$ (4)
2.1.3	$T_{20}=3(20)^2-8(20)+7$ $=1047$	✓substitution ✓answer (2)
2.2	$T_n=-7n+42$ $-7n+42=-140$ $-7n=-182$ $n=26$	✓ $T_n=-7n+42$ ✓ $-7n+42=-140$ ✓ $n=26$ (3)
2.3	$S_n=\frac{n}{2}(a+l)$ **OR/OF** $S_n=\frac{n}{2}[2a+(n-1)d]$ $S_n=\frac{n}{2}(35-7n+42)$ $\quad S_n=\frac{n}{2}(70-7n+7)$ $S_n=\frac{n}{2}(-7n+77)$ $S_n=-\frac{7}{2}n^2+\frac{77}{2}n$ $-\frac{7}{2}n^2+\frac{77}{2}n=3n^2-8n+7$ $13n^2-93n+14=0$ $(n-7)(13n-2)=0$ $n=7$ *or* $n=\frac{2}{13}$ NA $\therefore n=7$	✓ $S_n=\frac{n}{2}(35-7n+42)$ or $S_n=\frac{n}{2}(70-7n+7)$ ✓simplification of S_n ✓equating ✓standard form ✓factors ✓answer with selection (6)
		[16]

Nov 2019 Solutions

2.1.1	209 ; 186	✓209 ✓186 (2)
2.1.2	321 ; 290 ; 261 ; 234 1st diff -31 -29 -27 2nd diff 2 2 $2a = 2$ $3a + b = -31$ $a + b + c = 321$ $a = 1$ $3(1) + b = -31$ $1 + (-34) + c = 321$ $b = -34$ $c = 354$ $T_n = n^2 - 34n + 354$	✓ 2^{nd} diff $= 2$ ✓ $a = 1$ ✓ $b = -34$ ✓ $c = 354$ (4)
2.1.3	$n^2 - 34n + 354 = 74$ $n^2 - 34n + 280 = 0$ $(n-14)(n-20) = 0$ $n = 14$ or $n = 20$	✓equating T_n to 74 ✓standard form ✓14 ✓ 20 (4)
2.1.4	$f^{/}(n) = 0$ $2n - 34 = 0$ $2n = 34$ $n = 17$ Term 17 will have the smallest value **OR/OF** $n = \frac{-b}{2a}$ $n = \frac{34}{2}$ $n = 17$ Term 17 will have the smallest value **OR/OF** $n = \frac{14+20}{2} = 17$ Term 17 will have the smallest value	✓ $2n - 34 = 0$ ✓answer (2) **OR/OF** ✓substitution ✓answer (2) **OR/OF** ✓substitution ✓answer (2)

2.2.1	$a=\frac{5}{8}$; $r=\frac{1}{2}$; $n=21$ $S_n=\frac{a(1-r^n)}{1-r}$ $S_{21}=\frac{\frac{5}{8}\left(1-\left(\frac{1}{2}\right)^{21}\right)}{1-\frac{1}{2}}$ $=1{,}2499...$ $=1{,}25$	✓ r ✓ substitution into the correct formula ✓ answer (3)
2.2.2	$T_n>\frac{5}{8192}$ $ar^{n-1}>\frac{5}{8192}$ $\frac{5}{8}\left(\frac{1}{2}\right)^{n-1}>\frac{5}{8192}$ $\left(\frac{1}{2}\right)^{n-1}>\frac{1}{1024}$ $\left(\frac{1}{2}\right)^{n-1}>\left(\frac{1}{2}\right)^{10}$ *or* $2^{-n+1}>2^{-10}$ $\therefore n-1<10$ $-n+1>-10$ $n<11$ $n<11$ $\therefore n=10$ $\therefore n=10$ **OR/OF** 8 ; 16 ; 32 ; … ; 8192 $8.2^{n-1}<8192$ $2^{n-1}<1024$ $2^{n-1}<2^{10}$ $n-1<10$ $n<11$ $\therefore n=10$	✓ substitution into the correct formula ✓ method /same base or log ✓calculating n ✓ answer **OR/OF** ✓ substitution into the correct formula ✓ method ✓calculating n ✓ answer (4)
		[19]

Feb-March 2018 Solutions

3.1	$-1\ ;\ 2\ ;\ 5$ $T_n = -1+(n-1)(3)$ $= 3n-4$	✓ $3n$ ✓ -4 (2)
3.2	$T_{43} = 3(43)-4$ $=125$ **OR/ OF** $T_{43} = -1+(43-1)(3)$ $=125$ NOTE: Answer only 2 / 2	✓ subs of 43 ✓ answer (2)
3.3	$T_n = 3n-4$ $S_n = \sum_{k=1}^{n} T_k = -1+2+5+\ldots\ldots\ldots+3n-4$ $S_n = \frac{n}{2}[-1+3n-4]$ or $S_n = \frac{n}{2}[-2+(n-1)3]$ $= \frac{n}{2}[3n-5]$ $= \frac{3n^2-5n}{2}$ **OR/*OF*** $T_n = 3n-4$ $\sum_{k=1}^{n} T_k = 3(1)-4+3(2)-4+3(3)-4+\ldots\ldots\ldots+3n-4$ $= 3(1+2+3+\ldots\ldots+n)-4n$ $= \frac{3n(n+1)}{2}-4n$ $= \frac{3n^2-5n}{2}$	✓ $S_n = \sum_{k=1}^{n} T_k$ ✓ substitution into correct formula ✓ $\frac{n}{2}[3n-5]$ or $\frac{3n^2-5n}{2}$ **OR/*OF*** ✓ $(1)-4+3(2)-4+3(3)-4+\ldots\ldots\ldots+3n-4$ ✓ $3(1+2+3+\ldots\ldots+n)-4n$ ✓ $\frac{3n^2-5n}{2}$ (3)
3.4	$T_{11} = (T_{11}-T_{10})+(T_{10}-T_9)+(T_9-T_8)+\ldots+(T_3-T_2)+(T_2-T_1)+T_1$ $125 = 29+26+23+\ldots 2+T_1$ $= \frac{10}{2}(29+2)+T_1$ $= 155+T_1$ $T_1 = -30$ NOTE: Answer only 1 / 6 If they only use $3n-4$ breakdown 0 / 6 **OR/*OF*** $T_n = an^2+bn+c$ $\therefore T_{11} = 121a+11b+c = 125$	✓✓ generating sum ✓ $29+26+23+\ldots 2$ ✓ $\frac{10}{2}(29+2)$ ✓ 155 ✓ -30 **OR/*OF***

	$T_n - T_{n-1} = an^2 + bn + c - \left[a(n-1)^2 + b(n-1) + c\right]$ $= an^2 + bn + c - an^2 + 2an - a - bn + b - c$ $= 2an + b - a$ $T_n - T_{n-1} = 3n - 4$ $2a = 3$ and $b - a = -4$ $a = \frac{3}{2}$ and $b = -\frac{5}{2}$ $121a + 11b + c = 125$ $121\left(\frac{3}{2}\right) + 11\left(-\frac{5}{2}\right) + c = 125$ $c = -29$ $T_n = \frac{3}{2}n^2 - \frac{5}{2}n - 29$ $T_1 = \frac{3}{2}(1)^2 - \frac{5}{2}(1) - 29$ $= -30$	✓ $121a + 11b + c = 125$ ✓ calculating $T_n - T_{n-1}$ in terms of a, b and c ✓ $a = \frac{3}{2}$ ✓ $b = -\frac{5}{2}$ ✓ $c = -29$ ✓ -30 (6) **[13]**

Feb-March 2017 Solutions

3.1.1	24	✓ 24 (1)
3.1.2	**6** **6** **9** **15** **0** **3** **6** **3** **3** $2a = 3$ $3a + b = 0$ $a + b + c = 6$ $a = \frac{3}{2}$ $b = -\frac{9}{2}$ $c = 9$ $T_n = \frac{3}{2}n^2 - \frac{9}{2}n + 9$ **OR/OF**	✓ $a = \frac{3}{2}$ ✓ $b = -\frac{9}{2}$ ✓ $c = 9$ ✓ $T_n = \frac{3}{2}n^2 - \frac{9}{2}n + 9$ (4)

	$T_n = T_1 + (n-1)d_1 + \frac{(n-1)(n-2)d_2}{2}$ $= 6 + (n-1)(0) + \frac{(n-1)(n-2)(3)}{2}$ $= 6 + \frac{n^2 - 3n + 2}{1}\left(\frac{3}{2}\right)$ $= 6 + \frac{3}{2}n^2 - \frac{9}{2}n + 3$ $= \frac{3}{2}n^2 - \frac{9}{2}n + 9$	✓formula ✓substitution ✓simplifying ✓ $T_n = \frac{3}{2}n^2 - \frac{9}{2}n + 9$ (4)

3.1.3	$\frac{3}{2}n^2 - \frac{9}{2}n + 9 = 3249$ $3n^2 - 9n + 18 = 6498$ $3n^2 - 9n - 6480 = 0$ $n^2 - 3n - 2160 = 0$ $(n+45)(n-48) = 0$ $n \neq -45$ *or* $n = 48$	✓ equating general term to 3249 ✓ standard form ✓ factors ✓ $n \neq -45$ or $n = 48$ (4)
3.2	$-1\ ;\ 2\sin 3x\ ;\ 5;\ \ldots\ldots$ $2\sin 3x + 1 = 5 - 2\sin 3x$ $4\sin 3x = 4$ $\sin 3x = 1$ $3x = 90^\circ$ $x = 30^\circ$	✓ $2\sin 3x + 1 = 5 - 2\sin 3x$ ✓ $\sin 3x = 1$ ✓ $3x = 90^\circ$ ✓ $x = 30^\circ$ (4) **[13]**

Nov 2018 Solutions

3.1	$r = \frac{1}{2}$ and $S_\infty = 6$ $S_\infty = \frac{a}{1-r}$ $6 = \frac{a}{1-\frac{1}{2}}$ $a = 3$	✓substitution ✓answer (2)

3.2	$T_n = ar^{n-1}$ $T_8 = 3\left(\frac{1}{2}\right)^7$ $T_8 = \frac{3}{128}$	✓✓ $T_8 = 3\left(\frac{1}{2}\right)^7$ (2)
3.3	$\sum_{k=1}^{n} 3(2)^{1-k} = 5{,}8125$ $3 + \frac{3}{2} + \frac{3}{4} + \ldots = 5{,}8125$ $S_n = \frac{a(1-r^n)}{1-r} = 5{,}8125$ $\frac{3\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 5{,}8125$ $6\left[1-\left(\frac{1}{2}\right)^n\right] = 5{,}8125$ $\left(\frac{1}{2}\right)^n = \frac{1}{32} = 0{,}03125$ $2^{-n} = 2^{-5}$ or $n\log\frac{1}{2} = \log\frac{1}{32}$ $n = 5$ $\quad$ $n = 5$	✓ $r = \frac{1}{2}$ ✓substitution ✓simplification ✓answer (4)

3.4	$\sum_{k=1}^{20} 3(2)^{1-k} = p$ $3 + \frac{3}{2} + \frac{3}{4} + \ldots\ldots + 3.2^{-19} = p$ $\sum_{k=1}^{20} 24(2)^{-k}$ $= 12 + 6 + 3 + \ldots\ldots + 24.2^{-20}$ $= 4\left(3 + \frac{3}{2} + \frac{3}{4} + \ldots\ldots + 3.2^{-19}\right)$ $= 4p$	✓expansion ✓ expansion ✓ answer (3)

	OR/OF $S_{20}=\dfrac{3\left(\left(\frac{1}{2}\right)^{20}-1\right)}{\frac{1}{2}-1}=6=p$ $S_{20}=\dfrac{12\left(\left(\frac{1}{2}\right)^{20}-1\right)}{\frac{1}{2}-1}=24$ $24=4\times 6=4p$	**OR/OF** ✓ substitution and answer ✓substitution and answer ✓$4p$ (3)
		[11]

Nov 2019 Solutions

3.1	$\sum_{y=3}^{10}\dfrac{1}{y-2}-\sum_{y=3}^{10}\dfrac{1}{y-1}$ $=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{8}\right)-\left(\frac{1}{2}+\frac{1}{3}+....+\frac{1}{8}+\frac{1}{9}\right)$ $=1-\frac{1}{9}$ $=\frac{8}{9}$	✓ $\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{8}\right)$ ✓ $\left(\frac{1}{2}+\frac{1}{3}+....+\frac{1}{8}+\frac{1}{9}\right)$ ✓answer (3)
3.2	$\left(\frac{1}{3}\times\frac{2}{3}\right)+\left(\frac{2}{3}\times\frac{2}{3}\right)+\left(1\times\frac{2}{3}\right)+....+\left(4\times\frac{2}{3}\right)$ $=\frac{2}{9}+\frac{4}{9}+\frac{2}{3}+....+\frac{8}{3}$ $a=\frac{2}{9}$ and $d=\frac{2}{3}-\frac{4}{9}=\frac{2}{9}$ $S_n=\frac{n}{2}[2a+(n-1)d]$ OR $S_n=\frac{n}{2}(a+l)$ $S_{12}=\frac{12}{2}\left[2\left(\frac{2}{9}\right)+(12-1)\frac{2}{9}\right]$ $S_{12}=\frac{12}{2}\left(\frac{2}{9}+\frac{8}{3}\right)$ $=\frac{52}{3}\text{m}^2$ $=\frac{52}{3}\text{m}^2$ $\therefore$ for both sides $=2\times\frac{52}{3}=\frac{104}{3}=34{,}67\text{m}^2$	✓✓a ✓d ✓ substitution into the correct formula ✓ answer ✓ answer for both sides (6)

[9]

SECTION 3

FUNCTIONS & INVERSES

EXERCISE 1 Solutions 50 marks

1.

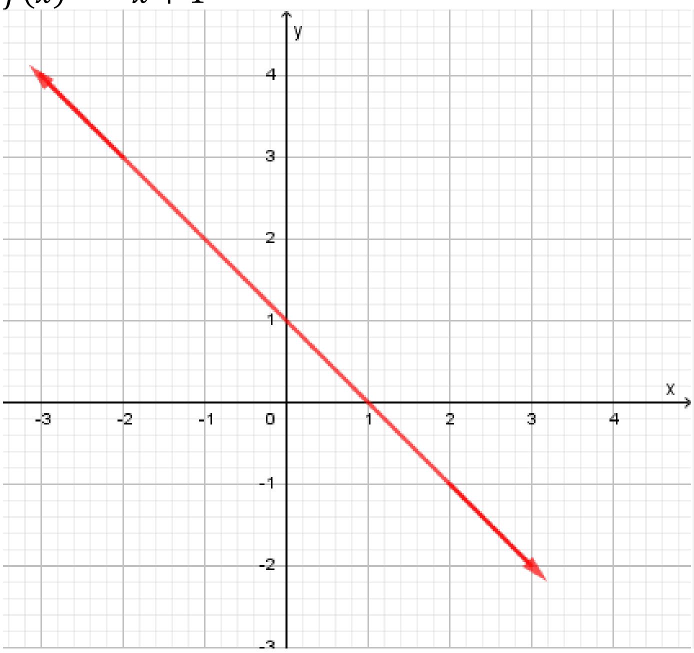
1.1 One – to – one function
1.2 Many – to – one function
1.3 One – to – many relation
1.4 Many – to – one function

1.5 One – to – one function	1.6 Not a function/ One – to – many relation
1.7 One – to – many relation	1.8 One – to – one function
1.9 Many – to – one function	1.10 Not a function/ One – to – many relation

(10)

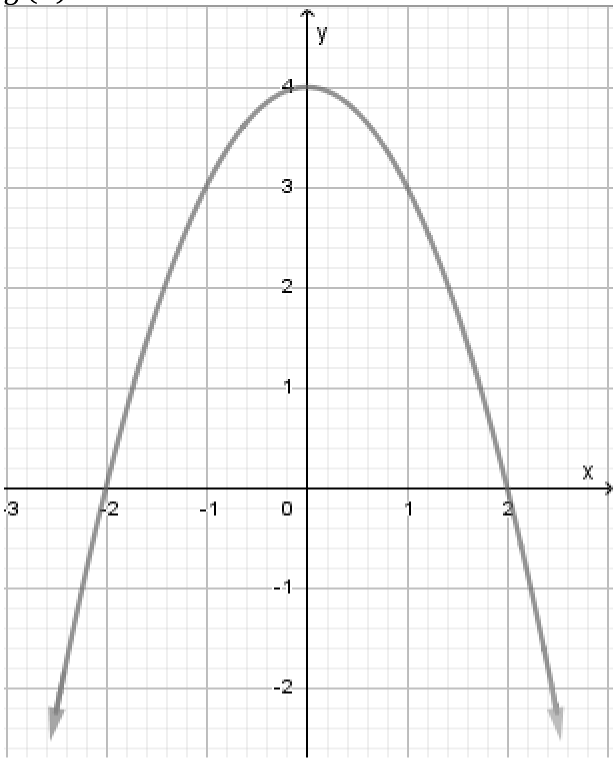
2. Sketch the following functions:

2.1 $f(x) = -x + 1$



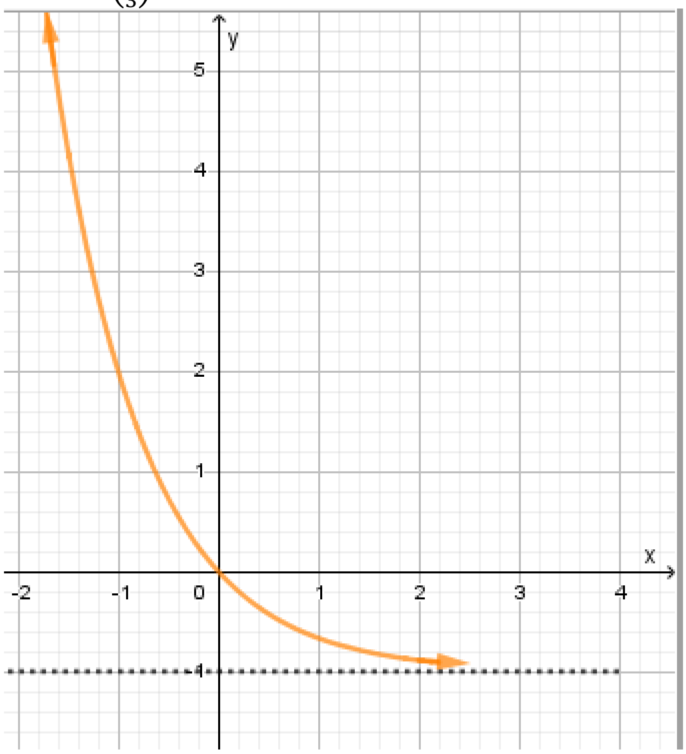
2

2.2 $g(x) = -x^2 + 4$

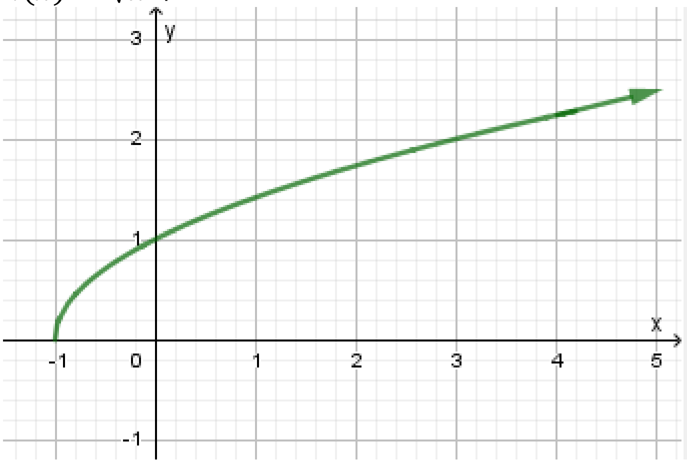
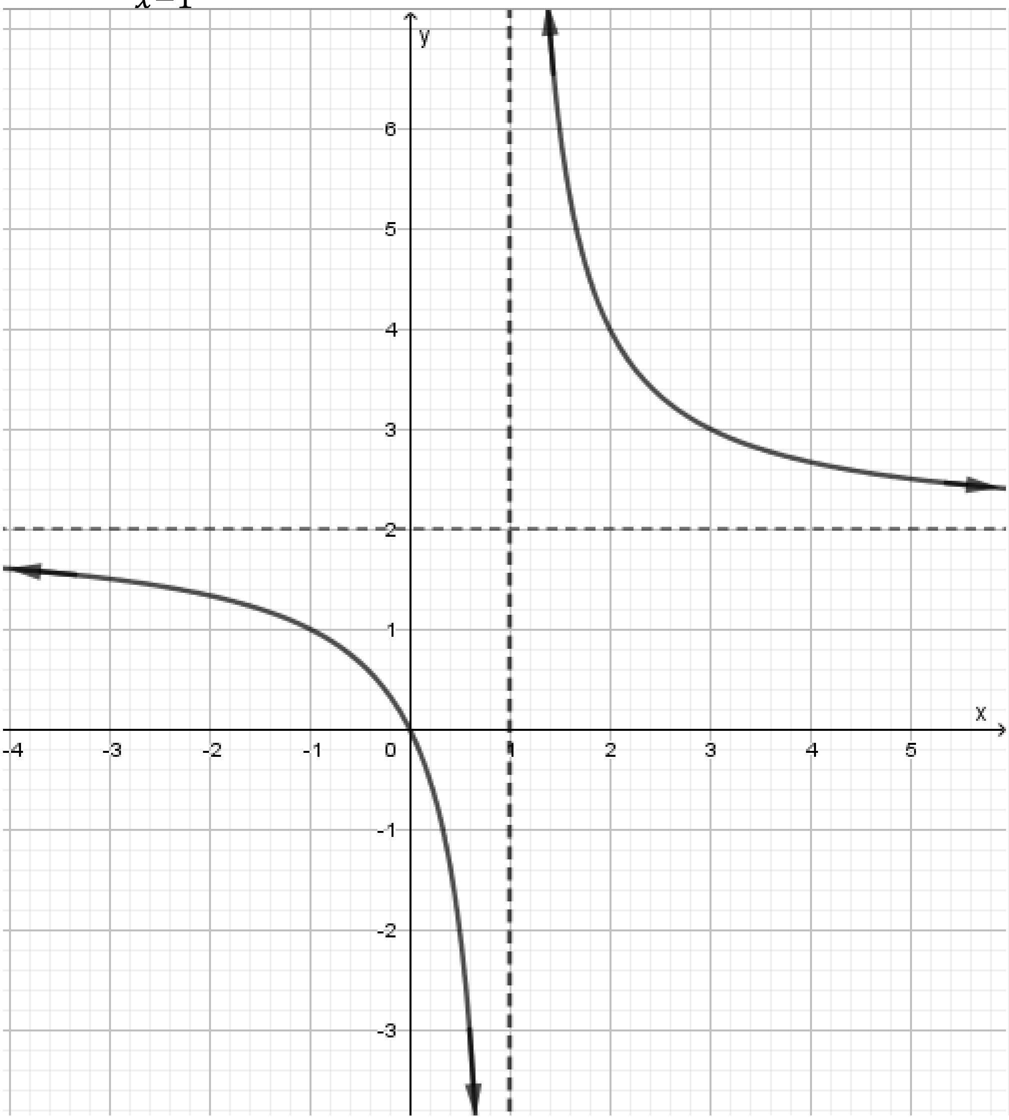


4

2.3 $h(x) = \left(\frac{1}{3}\right)^x - 1$



4

2.4 $k(x) = \sqrt{x+1}$  3
2.5 $j(x) = \frac{2}{x-1} + 2$  4

(17)

3.

$f(x) = -x + 1$	DOMAIN $x \in \mathbb{R}$	RANGE $y \in \mathbb{R}$
$g(x) = -x^2 + 4$	DOMAIN $x \in \mathbb{R}$	RANGE $y \leq 4$
$h(x) = \left(\frac{1}{3}\right)^x - 1$	DOMAIN $x \in \mathbb{R}$	RANGE $y > -1$
$k(x) = \sqrt{x+1}$	DOMAIN $x \geq -1$	RANGE $y \geq 0$
$j(x) = \frac{2}{x-1} + 2$	DOMAIN $x \in \mathbb{R}\,;\, x \neq 1$	RANGE $y \in \mathbb{R}\;;\, y \neq 2$

(10)

4.

$f'(x) = -(x-1) + 1 - 2$ $f'(x) = -x$
$g'(x) = -(x-1)^2 + 4 - 2$ $g'(x) = -x^2 + 2x + 1 \; OR \; g'(x) = -(x-1)^2 + 2$
$h'(x) = \left(\frac{1}{3}\right)^{x-1} - 3$
$k'(x) = \sqrt{x} - 2$
$j'(x) = \frac{2}{x-2}$

(10)

5. The vertical line cuts it once, every x- value is associated with only one y- value. (3)

TOTAL: 50

EXERCISE 2 **Solutions 56 marks**

1.1 $f(x) = \frac{1}{2}x - 3$

$$y = \frac{1}{2}x - 3$$
$$x = \frac{1}{2}y - 3$$
$$y = 2x + 6$$
$$f^{-1}(x) = 2x + 6$$

1.2 $j(x) = -2x^2 + 2$

$$y = -2x^2 + 2$$
$$x = -2y^2 + 2$$
$$y = \pm\sqrt{\frac{-x}{2} + 1}$$
$$j^{-1}(x) = \pm\sqrt{\frac{-x}{2} + 1}$$

1.3 $m(x) = 2^{-x} - 2$

$$y = 2^{-x} - 2$$
$$x = 2^{-y} - 2$$
$$-y = \log_2(x + 2)$$
$$y = -\log_2(x + 2)\ OR\ y = \log_2(x + 2)^{-1}\ OR\ \ y = \log_2\frac{1}{(x + 2)}$$
$$m^{-1}(x) -\log_2(x + 2)\boldsymbol{OR}\quad m^{-1}(x) = \log_2(x + 2)^{-1}\boldsymbol{OR}\quad m^{-1}(x) = \log_2\frac{1}{(x + 2)}$$

1.4 $n(x) = \frac{1}{x+1} - 2$

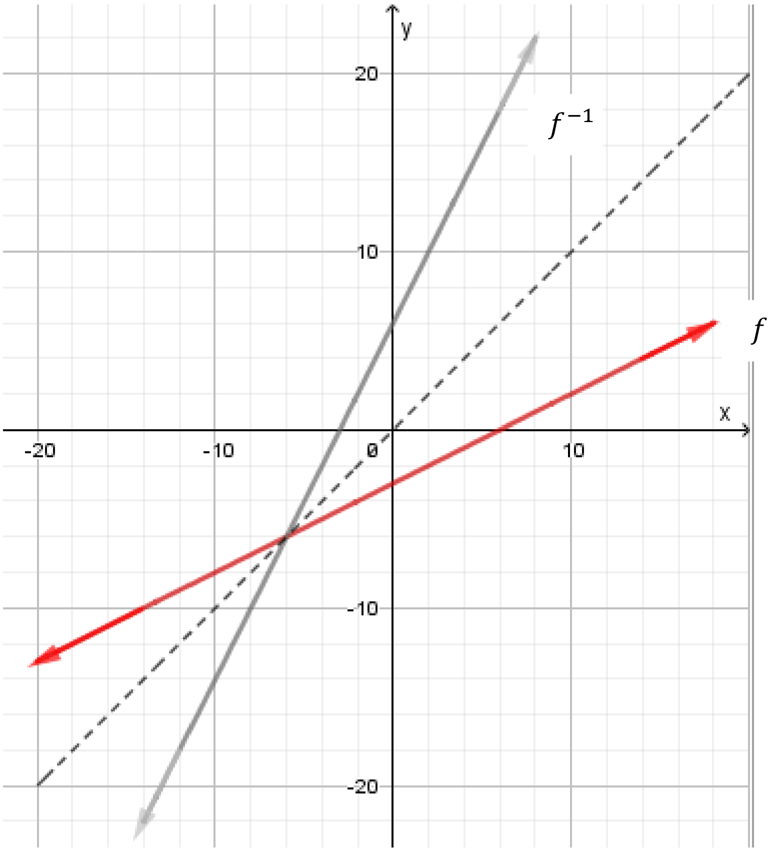
$$y = \frac{1}{x + 1} - 2$$
$$x = \frac{1}{y + 1} - 2$$
$$y + 1 = \frac{1}{x + 2}$$
$$y = \frac{1}{x + 2} - 1$$

$$n^{-1}(x) = \frac{1}{x + 2} - 1$$

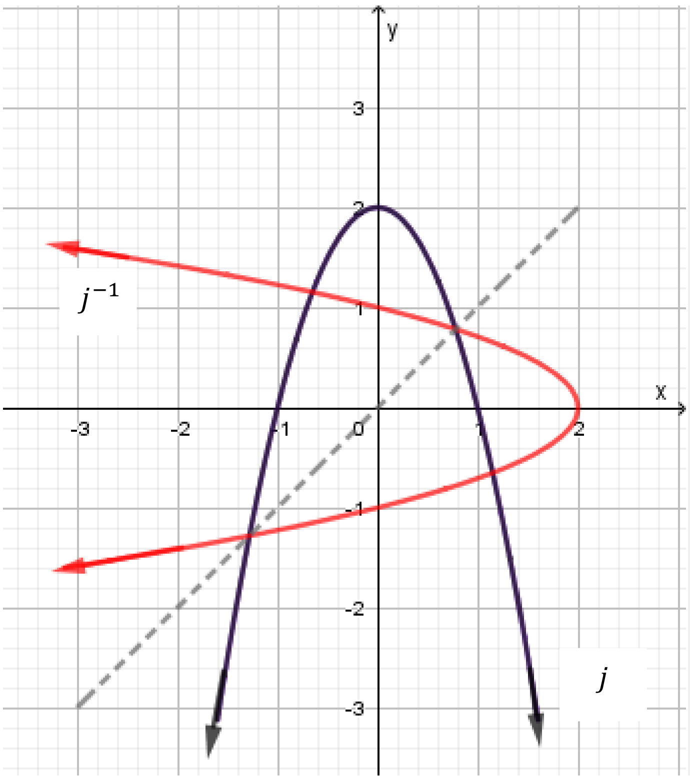
(8)

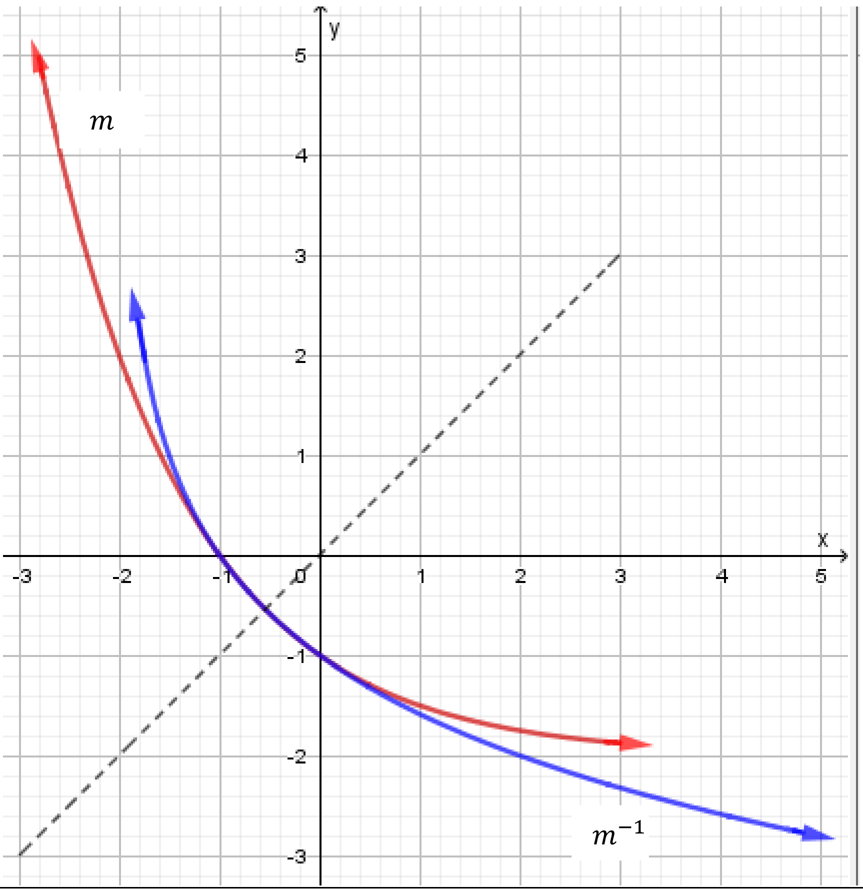
2 **(4 marks each function/4 marks each inverse)**

2.1

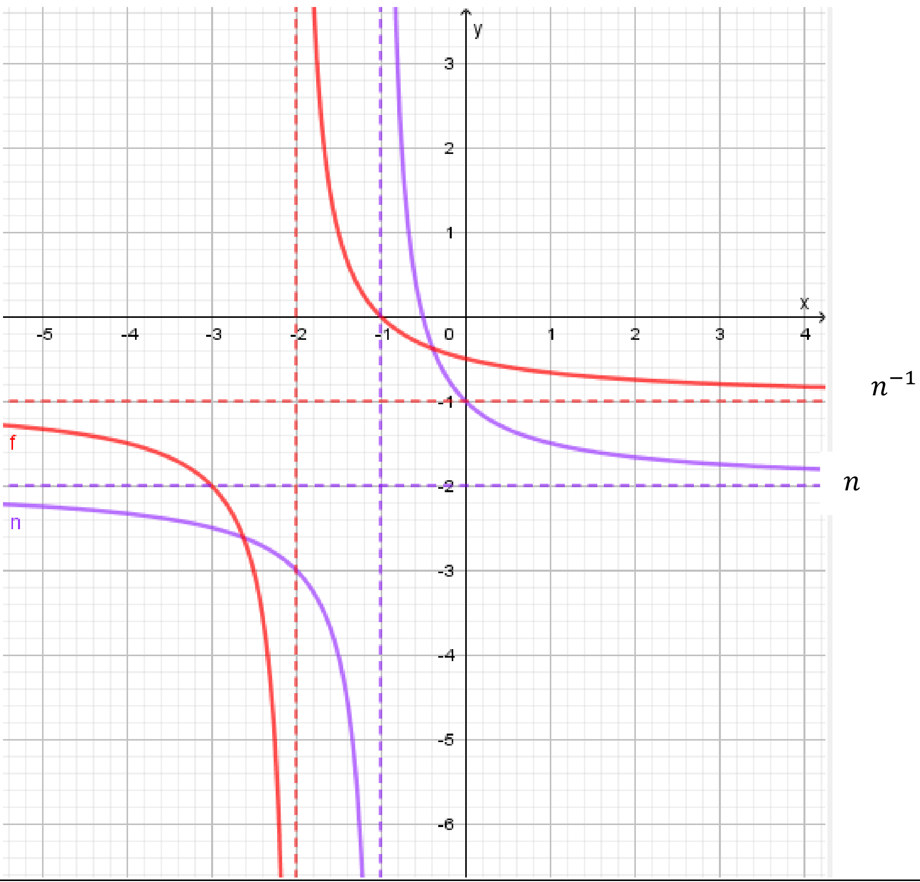


2.2





2.3



(32)

3 **(2 marks each function/ 2 marks each inverse)**

	Function		Inverse	
	Domain: $x \in \mathbb{R}$	Range: $y \in \mathbb{R}$	Domain: $x \in \mathbb{R}$	Range: $y \in \mathbb{R}$
3.2	Domain: $x \in \mathbb{R}$	Range: $y \leq 2$	Domain: $x \leq 2$	Range: $y \in \mathbb{R}$
3.3	Domain: $x \in \mathbb{R}$	Range: $y > -2$	Domain: $x > -2$	Range: $y \in \mathbb{R}$
3.4	Domain: $x \in \mathbb{R}\,;x \neq -1$	Range: $y \in \mathbb{R}\,;y \neq -2$	Domain: $x \in \mathbb{R}; x \neq -2$	Range: $y \in \mathbb{R};\ y \neq -1$

(16)

TOTAL: 56

Nov 2018 Solutions

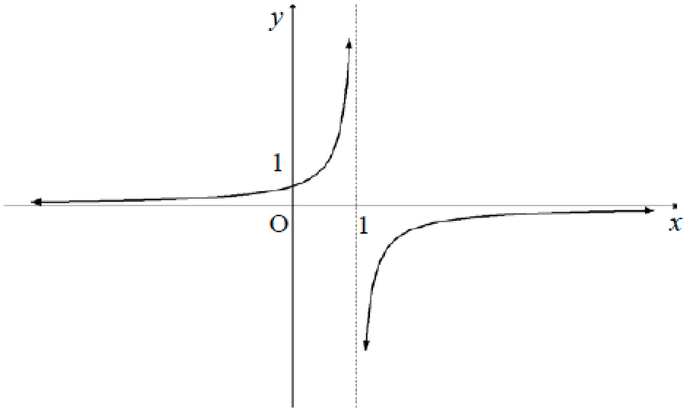
4.1	Yes For every x-value there is only one corresponding y value **OR/OF** One to one mapping (vertical line test)	✓answer ✓reason (2)
4.2	R(–12 ; –6)	✓answer (1)
4.3	$f(x) = ax^2$ substitute $(-6\,;-12)$ $-12 = a(-6)^2$ $a = \frac{-1}{3}$	✓substitution ✓answer (2)
4.4	$f : y = -\left(\frac{1}{3}\right)x^2$ $f^{-1} : x = -\left(\frac{1}{3}\right)y^2$ $y^2 = -3x$ $y = \pm\sqrt{-3x}$ Only $y = -\sqrt{-3x}$ and $x \leq 0$	✓swapping x and y ✓ $y^2 = -3x$ ✓ $y = -\sqrt{-3x}$ (3)
		[8]

Nov 2019 Solutions

4.1	$p=-1$	✓ $p=-1$ (1)
4.2	$y=\dfrac{a}{x-1}$ $-3=\dfrac{a}{0-1}$ $a=3$ $y=x^2+bx-3$ $0=(1)^2+(1)b-3$ $b=2$	✓coordinates D(0 ; –3) ✓substitute (0 ; –3) ✓substitute (1 ; 0) (3)
4.3	$y=x^2+2x-3$ axis of sym: $x=\dfrac{-b}{2a}$ $x=\dfrac{-2}{2(1)}$ $x=-1$ $y=(-1)^2+2(-1)-3=-4$ $C(-1;-4)$ **OR/OF** $\dfrac{dy}{dx}=0$ $2x+2=0$ $x=-1$ $y=(-1)^2+2(-1)-3=-4$ $C(-1;-4)$	✓substitution ✓ $x=-1$ ✓substitution ✓$y=-4$ (4) **OR/OF** ✓derivative ✓ $x=-1$ ✓substitution ✓$y=-4$ (4)

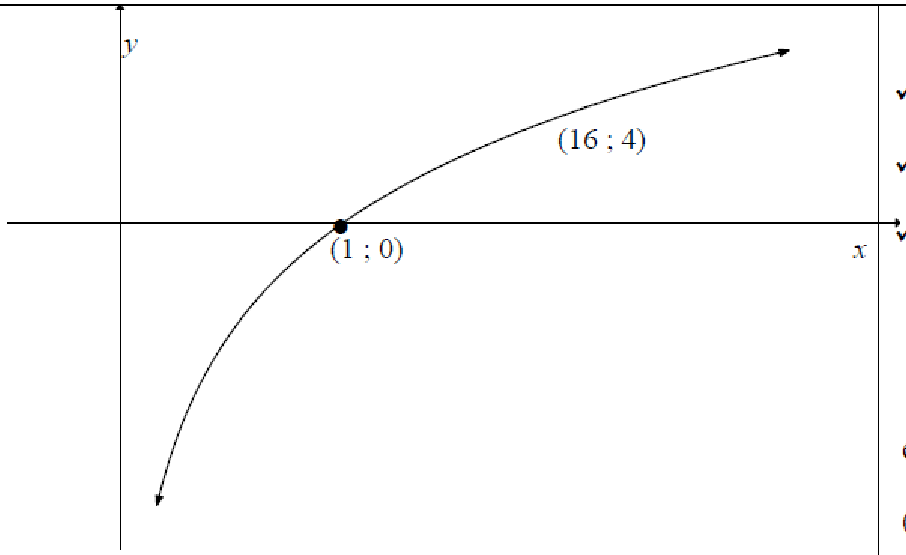
4.4	$y \in [-4;\infty)$ or $y \geq -4$	✓ -4 ✓ answer (2)
4.5	$m = \tan 45° = 1$ $y = mx + c$ $-4 = (1)(-1) + c$ $c = -3$ $y = x - 3$	✓gradient ✓subs m and $(-1\,;-4)$ ✓equation (3)
4.6	No, the line passes through C and D **OR/OF** No, a tangent through turning point C will have a gradient of 0	✓ No ✓ reason (2) **OR/OF** ✓ No ✓ reason (2)
4.7	$f(m-x) = f[-(x-m)]$ f is reflected in the y-axis and translated 1 unit to the left and 4 units upwards. Therefore: $m = -1$ $q = 4$ **OR/OF** Substitute $x = 0$ and $q = 4$ for one x- intercept $h(x) = (m-x)^2 + 2(m-x) - 3 + q$ $h(0) = (m-0)^2 + 2(m-0) - 3 + 4$ $0 = m^2 + 2m + 1$ $0 = (m+1)^2$ $m = -1$ $q = 4$	✓✓value of m ✓✓ value of q (4) **OR/OF** ✓✓value of m ✓✓ value of q (4)
		[19]

Nov 2018 Solutions

5.1	Domain: $x \in R\ ;\ x \neq 1$ **OR/OF** $x \in (-\infty;1) \cup (1;\infty)$	✓answer (1)
5.2	$x = 1$ $y = 0$	✓$x = 1$ ✓$y = 0$ (2)
5.3		✓ y intercept ✓vertical asymptote ✓shape (3)
5.4	$x \geq 0\ ;\ x \neq 1$ **OR/OF** $0 \leq x < 1$ or $x > 1$ **OR/OF** $x \in [0;1) \cup (1;\infty)$	✓$x \geq 0$ ✓$x \neq 1$ (2) **OR/OF** ✓ $0 \leq x < 1$ ✓ $x > 1$
		[8]

Nov 2019 Solutions

5.1	$f(x) = k^x$ $16 = k^4$ $k = 2$	✓substitution $(4\ ;\ 16)$ ✓answer (2)
5.2	$f:\quad y = 2^x$ $f^{-1}:\quad x = 2^y$ $y = \log_2 x$	✓ $x = 2^y$ ✓ $y = \log_2 x$ (2)

5.3		✓asymptote ✓shape ✓✓for any two valid points eg.(16 ; 4) or (2 ; 1) or (4 ; 2) or (1 ; 0) (4)
5.4.1	$x \in (1\,;\infty)$ or $x > 1$	✓1 ✓ answer (2)
5.4.2	$0 < x \leq \frac{1}{2}$ or $x \in \left(0;\frac{1}{2}\right]$	✓ $\frac{1}{2}$ ✓ answer (2)
5.5	$2^x - 2^{-x} = \frac{15}{4}$ $2^x - \frac{1}{2^x} = \frac{15}{4}$ $2^{2x} - 1 = \frac{15}{4} \times 2^x$ $4.2^{2x} - 4 = 15 \times 2^x$ $4.2^{2x} - 15.2^x - 4 = 0$ $(4.2^x + 1)(2^x - 4) = 0$ $4.2^x + 1 = 0$ or $2^x - 4 = 0$ $2^x = \frac{-1}{4}$ or $2^x = 2^2$ N/A $\quad x = 2$	✓ $2^x - 2^{-x} = \frac{15}{4}$ ✓standard form ✓factors ✓answer (4)

	OR/OF	OR/OF
	$2^x - 2^{-x} = \frac{15}{4}$	✓ $2^x - 2^{-x} = \frac{15}{4}$
	$2^x - \frac{1}{2^x} = \frac{15}{4}$	
	Let $k = 2^x$	
	$k^2 - 1 = \frac{15}{4} \times k$	
	$4.k^2 - 4 = 15 \times k$	
	$4.k^2 - 15.k - 4 = 0$	✓standard form
	$(4.k + 1)(k - 4) = 0$	✓factors
	$k = \frac{-1}{4}$ or $k = 4$	
	$2^x = \frac{-1}{4}$ or $2^x = 2^2$	
	N/A $\quad x = 2$	✓answer (4)
		[16]

SECTION 4

FINANCIAL MATHEMATICS

EXERCISE 1 Solutions

1.1	$A = P(1 - in)$ $A = 150000(1 - 0{,}12 \times 5)$ $A = R60\,000$
1.2	$A = P(1 - i)^n$ $A = 15000(1 - 0{,}12)^5$ $A = R79\,159{,}78$
2.	$A = (1 + i)^n$ $A = 12000(1 + \frac{0{,}09}{12})^{120}$ $A = R29\,416{,}28$
3.	$A = P(1 + i)^n$ $A = 15\,500\left(1 + \frac{0{,}1}{4}\right)^8\left(1 + \frac{0{,}085}{12}\right)^{36}\left(1 + \frac{0{,}11}{2}\right)^4$ $A = R30\,163{,}88$
4.	$A = P(1 + i)^n$ $20000\left(1 + \frac{0{,}09}{4}\right)^{20}(1 + 0{,}1)^3 + x\left(1 + \frac{0{,}09}{4}\right)^8(1 + 0{,}1)^3 - 12000(1 + 0{,}1)^3 = 43\,062{,}27$ $x = R11\,000$
5.	$A = P(1 + i)^n$ $\frac{A}{(1 + i)^n} = P$ $P = A(1 + i)^{-n}$ calculating back in time $50\,000 = 20000\left(1 + \frac{0{,}09}{12}\right)^{-12} + 25000\left(1 + \frac{0{,}095}{2}\right)^{-2}\left(1 + \frac{0{,}09}{12}\right)^{-24}$ $+ x\left(1 + \frac{0{,}095}{2}\right)^{-4}\left(1 + \frac{0{,}09}{12}\right)^{-24}$ $A = R18\,252{,}73$

Nov 2018 Solutions

7.1.1	$F=\frac{x[(1+i)^n-1]}{i}$ $F=\frac{15\ 000\left[\left(1+\frac{0,088}{4}\right)^{16}-1\right]}{\frac{0,088}{4}}$ $F\quad = \text{R}283\ 972{,}28$	✓ $\frac{0,088}{4}$ and $n=16$ ✓substitution into correct formula ✓answer (3)
7.1.2	$\text{A}=\text{R}283\ 972{,}28-100\ 000\left(1+\frac{0{,}088}{4}\right)^4$ $=\text{R}\ 174\ 877{,}60$	✓ future value – amount including interest ✓ $100\ 000\left(1+\frac{0{,}088}{4}\right)^4$ ✓answer (3)
	OR/OF Amount at end of 3 years: $F=\frac{15\ 000\left[\left(1+\frac{0,088}{4}\right)^{12}-1\right]}{\frac{0,088}{4}}-100\ 000$ $=\text{R}103\ 459{,}12$ Amount at end of 4 years: $P(1+i)^n\ +\ \frac{x[(1+i)^n-1]}{i}$ $=103\ 459{,}12\left(1+\frac{0{,}088}{4}\right)^4+\frac{15000\left[\left(1+\frac{0{,}088}{4}\right)^4-1\right]}{\frac{0{,}088}{4}}$ $=\text{R}\ 174\ 877{,}60$	**OR/OF** ✓ R15 000 including interest – R100 000 ✓ $\left(1+\frac{0{,}088}{4}\right)^4$ on P and x in F_v ✓method (3)
7.2.1	$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$ $1\ 500\ 000=\frac{x\left[1-\left(1+\frac{0{,}105}{12}\right)^{-12\times 20}\right]}{\frac{0{,}105}{12}}$ $x=\text{R}14\ \ 975{,}70$	✓ $i=\frac{0{,}105}{12}$ ✓ $n=240$ ✓ substitution into correct formula ✓ answer (4)

7.2.2	$P=\frac{x[1-(1+i)^{-n}]}{i}$ $P=\frac{14\ 975{,}70\left[1-\left(1+\frac{0{,}105}{12}\right)^{-12\times 8}\right]}{\frac{0{,}105}{12}}$ $P=\text{R}969\,927{,}74$ **OR/OF** Balance outstanding = A – F $=1\ 500\ 000\left(1+\frac{0{,}105}{12}\right)^{144}-\frac{14\ 975{,}70\left[\left(1+\frac{0{,}105}{12}\right)^{144}-1\right]}{\frac{0{,}105}{12}}$ $=\text{R}5\ 259\ 229{,}61-\text{R}4\ 289\ 302{,}47$ $=\text{R}969\ 927{,}14$	✓R14 975,70 in P_v-formula ✓✓ $n=96$ ✓ substitution into correct formula ✓ answer (5) **OR/OF** ✓ $n=144$ in A-formula ✓ $n=144$ in F_v-formula ✓ R14 975,70 ✓ A – F ✓ answer (5)
		[15]

Nov 2019 Solutions

6.1	Kuda : $A=P(1+in)$ $=5\ 000(1+0{,}083\times 4)$ $=R6\ 660{,}00$ Final Answer : R6 660,00 + R266,40 $=R6\ 926{,}40$ **OR/OF** Kuda : $A=P(1+in)\times 1{,}04$ $=5\ 000(1+0{,}083\times 4)\times 1{,}04$ $=R6\,926{,}40$ Thabo : $A=P(1+i)^n$ $=5\ 000\left(1+\frac{0{,}081}{12}\right)^{12\times 4}$ $=R6\,905{,}71$ Kuda will have a better investment	✓substitution into the correct formula ✓final answer **OR/OF** ✓substitution into the correct formula ✓final answer ✓substitution into the correct formula ✓answer ✓conclusion (5)

6.2.1	$P=\dfrac{x[1-(1+i)^{-n}]}{i}$ $525\ 000=\dfrac{6\ 000\left[1-\left(1+\frac{0,1}{12}\right)^{-n}\right]}{\frac{0,1}{12}}$ $\dfrac{35}{48}=1-\left(1+\dfrac{0,1}{12}\right)^{-n}$ $-n\log\left(1+\dfrac{0,1}{12}\right)=\log\dfrac{13}{48}$ $-n=\dfrac{\log\frac{13}{48}}{\log\left(1+\frac{0,1}{12}\right)}$ $n=157,40$ $n=158$ payments	✓ $\dfrac{0,1}{12}$ ✓substitution into the correct formula ✓simplification ✓use of logs ✓answer (5)

6.2.2	Difference: R6 000 – R5 066,36 = R933,64 $F=\dfrac{x[(1+i)^{n}-1]}{i}$ $F=\dfrac{933,64\left[\left(1+\frac{0,1}{12}\right)^{108}-1\right]}{\frac{0,1}{12}}$ $=R162\,503,51$ **OR/OF**	✓ R933,64 ✓ $n=108$ ✓ substitution into the correct formula ✓ answer (4) **OR/OF**

[14]

Exercise 2 Solutions

1.1	$\left(1+\frac{i^m}{m}\right)^m = 1+i_e$ $\left(1+\frac{i^{12}}{12}\right)^{12} = 1{,}083$✓ $\frac{i^{12}}{12} = \left(\sqrt[12]{1.083}-1\right)$✓ $i^{12} = 0{,}08$ $r = 8\%$✓	✓Substitution ✓Simplification ✓Answer (3)
1.2.1	$Int = R8696{,}97 \times 60 = R121\,818{,}20$✓	Answer (1)
1.2.2	$400000 = \frac{9000\left[1-\left(1+\frac{11}{1200}✓\right)^{-n}\right]}{\frac{11}{1200}}$✓ $\left(1+\frac{11}{1200}\right)^{-n} = \frac{16}{27}$ $n = -\frac{\log\frac{16}{27}}{\log\left(1+\frac{11}{1200}\right)}$✓ $= 57{,}34284187 =\approx 57$ full payments✓	✓$\frac{11}{1200}$ ✓Substitution into correct formula ✓Use of logs ✓Answer (4)
1.2.3	$y\left(1+\frac{11}{1200}\right)^{-58✓} = 400000 - \frac{9000\left[1-\left(1+\frac{11}{1200}\right)^{-57✓}\right]}{\frac{11}{1200}}$✓ $= 3094{,}83261 \approx 3094{,}83$ ✓	✓✓n=57, 58 ✓Substitution ✓Answer (4)
1.2.4	$Int = 121\,818{,}20 - \{[(9000 \times 57) + 3094{,}83]$✓ $-400\,000\}$ $= 5723{,}37$ ✓is saved	✓Simplification ✓Answer (2)
		[14]

2.1		$A = P(1+i)^n$ $P = \frac{A}{(1+i)^n}$ $= \frac{8450}{\left(1+\frac{0{,}12}{12}\right)^{120}}$ $= R2560{,}31$	✓ ✓ ✓	making P subject of formula correct substitution answer	(3)

2.2		Let the original amount be Q $A = P(1-i)^n$ $\frac{1}{2}Q = Q(1-0,047)^n$ $\frac{1}{2} = (1-0,047)^n$ $\log\frac{1}{2} = n\log(1-0,047)$ $n = 14,40\ years/jare$	✓ ✓ ✓	subst. logs answer	(3)
2.3	2.3.1	The scrap value of the tractor is R$x(1-0,2)^5$	✓	scrap value	(1)
	2.3.2	so Price of new tractor will be R $x(1+0,18)^5$ after 5 years	✓	Price of new tractor	(1)
	2.3.3	$F = \frac{x[(1+i)^n - i]}{i}$ $F = \frac{8\,000\left[\left(1+\frac{0,10}{12}\right)^{60} - 1\right]}{\frac{0,10}{12}}$ $= R619\,496,58$ NB: i *and* n only get marks when used in an annuity formula	✓ ✓ ✓ ✓	$i = \frac{0,10}{12}$ $n = 60$ subst. into an annuity formula answer	(4)
	2.3.4	Sinking fund = New tractor price – Scrap value $R619\,496,58 = \text{R}\,x(1+0,18)^5 - \text{R}x(1-0,2)^5$ $= \text{R}x(1+0,18)^5 - (1-0,2)^5)$ $R619\,496,58 = x(1+0,18)^5 - (1-0,2)^5)$ $x = \frac{R619\,496,58}{[(1+0,18)^5-(1-0,2)^5)]}$ $x = \text{R}316\,057.15$	✓ ✓ ✓ ✓	$R619\,496,58 =$ $x(1+0,18)^5 - (1-0,2)^5)$ factorising answer	(4)

[16]

3.1	$1+i_{eff}=\left(1+\frac{i_{nom}}{m}\right)^m$ $\left(1+\frac{i_m}{12}\right)^{12}=\left(1+\frac{0,13}{4}\right)^4$ $=1,136475928...$ $1+\frac{i_m}{12}=1,010718046...$ $\frac{i_m}{12}=0,0107180464...$ $i_m=0,1286165568...$ $\therefore r_m=12,86\%$ p.a. compounded monthly	✓ $\left(1+\frac{i_m}{12}\right)^{12}$ ✓ $\left(1+\frac{0,13}{4}\right)^4$ ✓ $1+\frac{i_m}{12}=1,010718046...$ (3)
3.2	$P_v=\frac{x[1-(1+i)^{-n}]}{i}$ $600\ 000=\frac{9\ 000\left[1-\left(1+\frac{0,1286165568}{12}\right)^{-n}\right]}{\frac{0,1286165568}{12}}$ $0,7145364267=1-\left(1+\frac{0,1286165568}{12}\right)^{-n}$ $\left(1+\frac{0,1286165568}{12}\right)^{-n}=1-0,7145364267$ $=0,2854635733$ $-n=\log_{\left(1+\frac{0,1286165568}{12}\right)}0,2854635733$ $=-117,5911312$ $n=117,59$ $\therefore$ 117 payments of $R9\ 000$	✓ substitution into the correct formula ✓ simplify RH ✓ correct log $-n=\log_{1,010718...}0,28546$ or $-n=\frac{\log 0,28546...}{\log 1,010718...}$ ✓ $n=117,59$ ✓ 117 payments (5)

3.3	$A = P(1+i)^n$ $= 600\ 000\left(1 + \frac{0,1286165568}{12}\right)^{117}$ $= R2\ 088\ 640.059$ $F_v = \frac{x\left[(1+i)^n - 1\right]}{i}$ $= \frac{9\ 000\left[\left(1 + \frac{0,1286165568}{12}\right)^{117} - 1\right]}{\frac{0,1286165568}{12}}$ $= R2\ 083\ 364,827$ Outstanding balance after 117 payments: $= R2\ 088\ 640,059 - R2\ 083\ 364,827$ $= R5\ 275,232$ Final payment $= 5\ 275,232\left(1 + \frac{0,1286165568}{12}\right)$ $= R5\ 331,77$	✓ substitution into the correct formula ✓ substitution into the correct formula ✓ subtract: $A - F$ ✓ interest of 1 month ✓ answer (5)
3.4	$117(R\,9\ 000) + R5\ 331,77$ $= R1\ 058\ 331,77$	✓ 117(9 000) ✓ $R1\ 058\ 331,77$ (2) **[15]**

4.1	$A = P(1+i)^n$ $30\ 440 = 12\ 500\left(1 + \frac{0,09}{4}\right)^{4k}$ $\frac{1522}{625} = \left(\frac{409}{400}\right)^{4k}$ OR $2,4352 = (1,0225)^{4k}$ $\therefore 4k = \log_{1,0225} 2,4352$ $\therefore 4k = 40,0002\ldots$ $\therefore k = 10$ years	✓substitution into correct formula *(A and P)* ✓$i = \frac{0,09}{4}$ OR $\frac{9}{400}$ ✓ $\therefore 4k = \log_{1,0225} 2,4352$ ✓answer	(4)
4.2.1	$1 + i^{eff} = (1 + \frac{i^n}{n})^n$ $i^{eff} = (1 + \frac{0,18}{12})^{12} - 1 = 1,195618\ldots\ldots - 1 = 0,195618\ldots.$ $i^{eff} = 19,56\%$	✓substitution into correct formula ✓$i = \frac{0,18}{12}$ ✓answer	(3)

4.2.2	$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$ $500\ 000=\frac{x\left[1-\left(1+\frac{0{,}18}{12}\right)^{-60}\right]}{\frac{0{,}18}{12}}$ $x=R12\ 696{,}71$	✓ substitution into correct formula ✓P = 500 000 ✓ -n = – 60 ✓answer	(4)
4.2.3	$\text{balance/ saldo}=500\ 000(1+\frac{0{,}18}{12})^{24}-\frac{12700[(1+\frac{0{,}18}{12})^{24}-1]}{\frac{0{,}18}{12}}$ $=714\ 751{,}406-363\ 645{,}7142$ $=R351105{,}69$ **OR**	✓substitute into correct formula ✓n = 24 or/of n = -36 ✓ $\frac{0{,}18}{12}$ ✓answer	
	OR $P=\frac{x\left[1-(1+i)^{-n}\right]}{i}=\frac{12700\left[1-\left(1+\frac{0{,}18}{12}\right)^{-36}\right]}{\frac{0{,}18}{12}}$ $=R351\ 290{,}69$		(4)
4.2.4	$500\ 000(1-i)^2=304\ 200$ $(1-i)^2=0{,}6094$ $i=22\%$	✓substitute into correct formula ✓answer	(2)
			[17]

5.1	$A = P(1-i)^n$ $= 500000(1-0{,}2)^6$ $=131\ 072$	✓substitution into correct formula ✓answer (2)
5.2	$A_e = P(1+i)^n$ $= 500000\,(1+0{,}07)^6$ $= 750\ 365{,}1759$ Cost of the new server: $750\ 365{,}18 - 131\ 072{,}00 = 619\ 293{,}18$	✓subst. into correct formula ✓ $= 750\ 365{,}1759$ ✓ $619\ 293{,}18$ (3)
5.3	$F_v = \left[\dfrac{x(1+i)^n - 1}{i}\right]$ $619\ 293{,}18 = \dfrac{x\left[\left(1+\dfrac{0{,}08}{12}\right)^{73} - 1\right]}{\dfrac{0{,}08}{12}}$ $x = 6\ 613{,}636652 \approx 6\ 613{,}64$	✓ $n = 12 \times 6 + 1$ ✓ $i = \dfrac{0{,}08}{12}$ ✓substitution into correct formula ✓answer (4)
5.4	$F_v = \dfrac{x\left[(1+i)^n - 1\right]}{i}$ $1\ 000\ 000 = \dfrac{15000\left[\left(1+\dfrac{0{,}08}{12}\right)^n - 1\right]}{\dfrac{0{,}08}{12}}$ $n = \log_{\frac{151}{150}}\left(\dfrac{1\ 000\ 000 \times \dfrac{1}{150}}{15000} + 1\right)$ $n = 55{,}3423758$ After 56 months there will be more than R 1 000 000 in the fund.	✓ $i = \dfrac{0{,}08}{12}$, ✓ correct substitution into formulae. ✓use of logs ✓ $n = 55{,}3423758$ ✓answer (5)
		[14]

SECTION 5

DIFFERENTIAL CALCULUS

EXERCISE 1 Solutions

1. $f'(x) = 2x + 3$
2. $f'(x) = 6x^2$
3. $f'(x) = -\frac{1}{x^2}$
4. $f'(x) = -\frac{2}{x^3}$

EXERCISE 2 Solutions

1. $\frac{dy}{dx} = 5 - \frac{12}{x^7}$
2. $\frac{1}{3}x^{-\frac{2}{3}} + 24x^3$
3. $f'(x) = 2ax + b$
4. $\frac{dV}{dr} = 4\pi r^2 - 2\pi$

Nov 2018 Solutions

8.1	$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$ $=\lim_{h\to0}\frac{x^2+2xh+h^2-5-x^2+5}{h}$ $=\lim_{h\to0}\frac{h(2x+h)}{h}$ $=\lim_{h\to0}(2x+h)$ $=2x$	✓ $x^2+2xh+h^2-5$ ✓simplification ✓factorisation ✓ $\lim_{h\to0}(2x+h)$ ✓ $2x$ (5)
	OR/OF $f(x+h)=(x+h)^2-5$ $=x^2+2xh+h^2-5$ $f(x+h)-f(x)=x^2+2xh+h^2-5-(x^2-5)$ $=2xh+h^2$ $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$ $=\lim_{h\to0}\frac{2xh+h^2}{h}$ $=\lim_{h\to0}\frac{h(2x+h)}{h}$ $=\lim_{h\to0}(2x+h)$ $=2x$	✓ $x^2+2xh+h^2-5$ ✓simplification ✓factorisation ✓ $\lim_{h\to0}(2x+h)$ ✓ $2x$ (5)
8.2.1	$y=3x^3+6x^2+x-4$ $\frac{dy}{dx}=9x^2+12x+1$	✓ $9x^2$ ✓ $12x$ ✓ 1 (3)
8.2.2	$y(x-1)=2x(x-1)$ $y=\frac{2x(x-1)}{x-1}$ if $x\neq1$ $y=2x$ $\frac{dy}{dx}=2$	✓ $y(x-1)$ ✓ $2x(x-1)$ ✓ $y=2x$ ✓answer (4)
		[12]

Nov 2019 Solutions

7.1	$f(x)=4-7x$ $f^{/}(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ $=\lim\limits_{h\to 0}\dfrac{4-7(x+h)-(4-7x)}{h}$ $=\lim\limits_{h\to 0}\dfrac{h(-7)}{h}$ $=-7$	✓ $4-7(x+h)$ ✓substitution ✓simplification ✓answer (4)
7.2	$y=4x^8+\sqrt{x^3}$ $=4x^8+x^{\frac{3}{2}}$ $\dfrac{dy}{dx}=32x^7+\dfrac{3}{2}x^{\frac{1}{2}}$	✓ $x^{\frac{3}{2}}$ ✓ $32x^7$ ✓ $\dfrac{3}{2}x^{\frac{1}{2}}$ (3)

7.3.1	$y=ax^2+a$ $\dfrac{dy}{dx}=2ax+0$ $\dfrac{dy}{dx}=2ax$	✓ $2ax$ (1)
7.3.2	$y=ax^2+a$ $\dfrac{dy}{da}=x^2+1$	✓ ✓ answer (2)

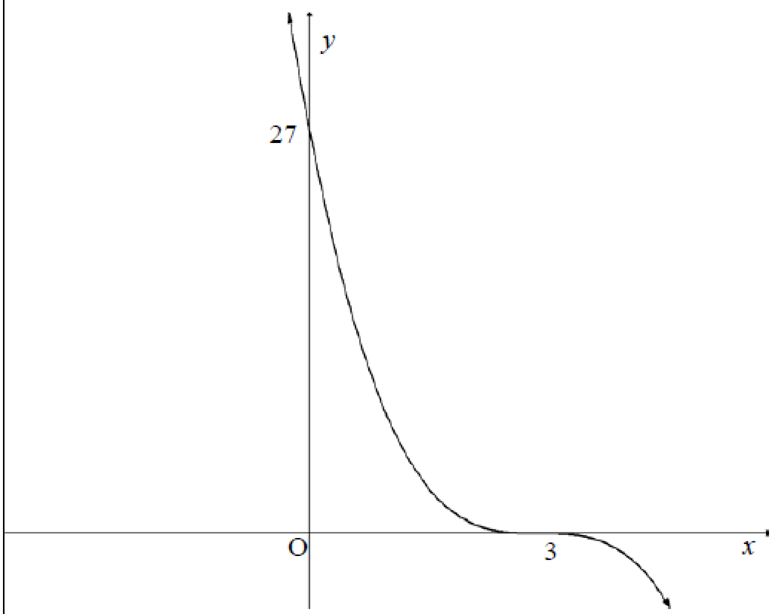
7.4	Substitute $(2\ ;\ b)$ in $y=x+\dfrac{12}{x}$ $b=2+\dfrac{12}{2}$ $b=8$ $m_{\text{tangent}}=\dfrac{dy}{dx}$ $\dfrac{dy}{dx}=1-\dfrac{12}{x^2}$ $m_{\text{tangent}}=1-\dfrac{12}{2^2}=-2$	✓ value of b ✓ $\dfrac{dy}{dx}=1-\dfrac{12}{x^2}$

	$m_{perp} = \frac{1}{2}$ Equation of perpendicular line: $y - y_1 = m(x - x_1)$ OR $y = mx + c$ $y - 8 = \frac{1}{2}(x-2)$ $8 = \frac{1}{2}(2) + c$ $y = \frac{1}{2}x + 7$ $c = 7$ $y = \frac{1}{2}x + 7$	✓gradient of perpendicular line ✓ equation (4)
		[14]

Nov 2018 Solutions

9.1.1	$g(x) = (x+5)(x-x_1)^2$ $20 = 5(x_1)^2$ $x_1^{\ 2} = 4$ $x_1 = 2$ $g(x) = (x+5)(x-2)^2$ $g(x) = (x+5)(x^2-4x+4)$ $g(x) = x^3 + x^2 - 16x + 20$	✓$(x+5)$ ✓repeated root ✓$x_1 = 2$ ✓ $g(x) = (x+5)(x^2-4x+4)$ (4)

9.1.2	$g(x) = x^3 + x^2 - 16x + 20$ $g'(x) = 3x^2 + 2x - 16$ $3x^2 + 2x - 16 = 0$ $(3x+8)(x-2) = 0$ $x = \frac{-8}{3}$ or $x = 2$ $R\left(\frac{-8}{3}; \frac{1372}{27}\right)$ or $R(-2{,}67; 50{,}81)$ $P(2;0)$	✓derivative ✓equating to zero ✓factors ✓co-ordinates of R ✓ co-ordinates of P (5)
9.1.3	$g''(x) = 6x + 2$ $g''(0) = 2$ $\therefore$ concave up **OR/OF** $g''(x) = 6x + 2$ $6x + 2 = 0$ $x = -\frac{1}{3}$ is the point of inflection $\therefore$ concave up	✓ $g''(x) = 6x + 2$ ✓ $g''(0) = 2$ ✓conclusion (3) **OR/OF** ✓ $g''(x) = 6x + 2$ ✓ $x = -\frac{1}{3}$ ✓conclusion (3)

9.2		✓ y – intercept of a cubic graph ✓ point of inflection and stationary point, $x = 3$ ✓ concave up for $x < 3$ and concave down for $x > 3$ (3)
		[15]

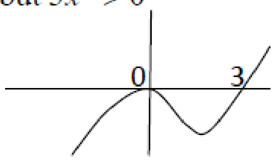
Nov 2019Solutions

8.1	36cm	✓answer (1)
8.2	$\therefore t = 6 \quad (-2t^2 + 3t - 6)$ have no real roots Insect reaches the floor only once.	✓✓✓ only once (3)
8.3	$h(t) = -2t^3 + 15t^2 - 24t + 36$ $h'(t) = -6t^2 + 30t - 24$ $-6t^2 + 30t - 24 = 0$ $t^2 - 5t + 4 = 0$ $(t-4)(t-1) = 0$ $t = 4$ or $t = 1$ Only $t = 4$ because maximum value required $h = -2(4)^3 + 15(4)^2 - 24(4) + 36 = 52\ cm$	✓ expansion ✓ $-6t^2 + 30t - 24 = 0$ ✓both values ✓answer (4)
		[8]

Nov 2019 Solutions

9.1	$f^{/}(x)=9x^2$ $3x^3=9x^2$ $3x^3-9x^2=0$ $3x^2(x-3)=0$ $x=0$ or $x=3$	✓ $f^{/}(x)=9x^2$ ✓ $x=0$ ✓ $x=3$ (3)
9.2.1	For f and $f^{/}$	✓ answer (1)
9.2.2	The point (0 ; 0) is : A point of inflection of f A turning point of $f^{/}$	✓ f: inflection point ✓ $f^{/}$: turning point (2)

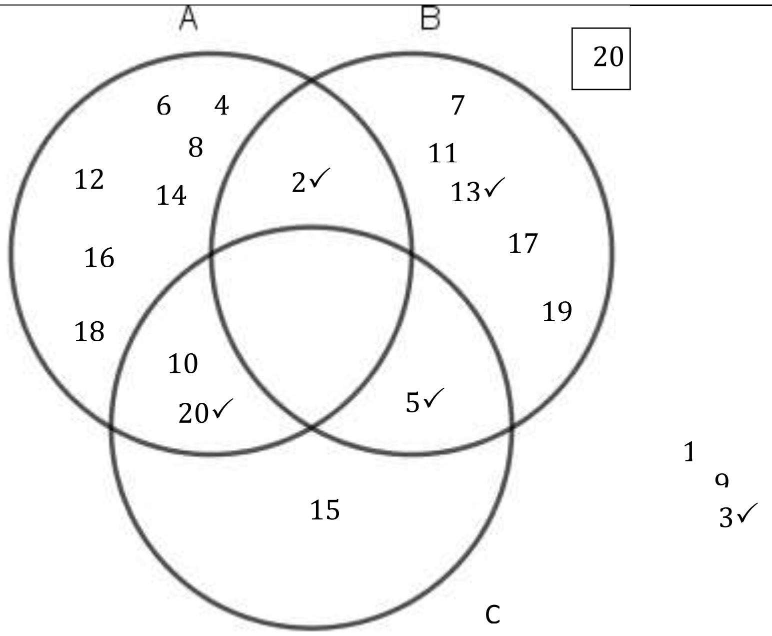
9.3	$f^{//}(x)=18x$
	$\text{Distance}=f^{//}(1)-f^{/}(1)$
	$=18(1)-9(1)^2$
	$=9$
		✓ $f^{//}(x)=18x$
		✓substitution
		✓answer (3)

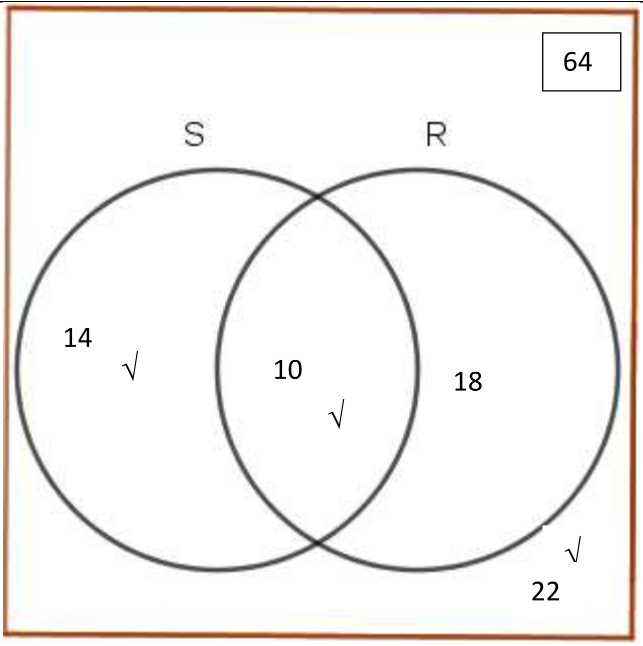
9.4	$3x^3-9x^2<0$ $3x^2(x-3)<0$ but $3x^2>0$  $\therefore x-3<0$ $\therefore x<3\ ,\ x\neq 0$	✓ $3x^3-9x^2<0$ ✓ factors ✓ $x<3$ ✓ $x\neq 0$ (4)
		[13]

SECTION 6

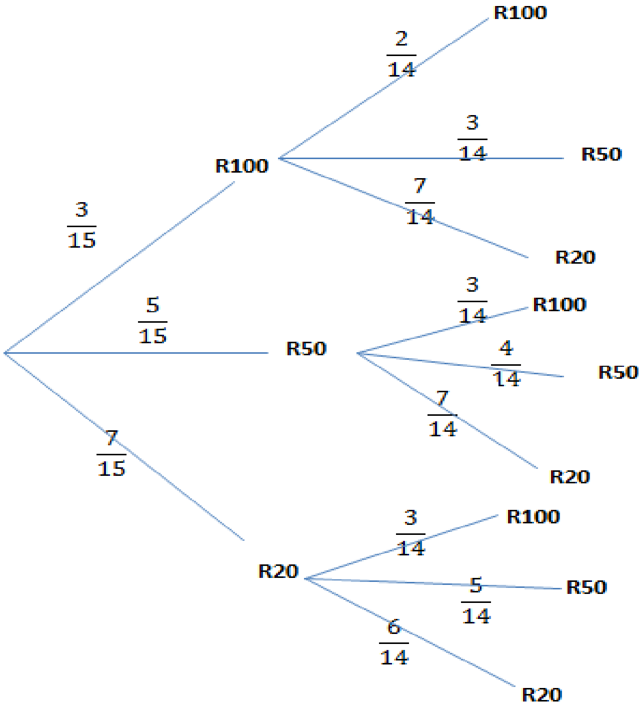
PROBABILITY & COUNTING PRINCIPLES

REVISION EXERCISE 1SOLUTIONS

QUESTION 1		
1.1		(5)
1.2 1.2.1	P(A)$= \frac{10}{20} = \frac{1}{2}\surd$	(1)
1.2.2	P′(B)$= 1 -$P(B) $= \frac{12}{20} = \frac{3}{5}\surd\surd$	(2)
1.2.3	P(A and B)$= \frac{1}{20}\surd$	(1)
1.2.4	P(A or B)$= \frac{16}{20} = \frac{4}{5}\surd\surd$	(2)
1.3	No; √since P(A or B or C)$\neq 1\surd$ A;B and C not exhaustive	(2)
		[14]

QUESTION2		
2.1 2.1.1		(3)
2.1.2		
a)	$\frac{10}{64} = \frac{5}{32}$ √√	(2)
b)	$\frac{32}{64} = \frac{1}{2}$√√	(2)
2.1.3	NO; √P(S and R)$\neq 0$√	(2)
2.2	$\frac{6}{30}$√√$= \frac{1}{5}$√	(3)
2.3	$1 - \left(\frac{10}{36} + \frac{1}{36}\right)$ $1 - \frac{11}{36}$√ $\frac{25}{36}$√√	(3)
		[15]

REVISION EXERCISE 2

SOLUTIONS		
Qn 1		
1.1		✓✓✓ branches with correct values
1.2.1	$\frac{3}{15}\times\frac{2}{14}=\frac{1}{35}\ or\ 0{,}03$	✓ answer
1.2.2	$\frac{3}{15}\times\frac{7}{14}+\frac{7}{15}\times\frac{3}{14}=\frac{1}{5}\ or\ 0{,}2$	✓product ✓answer
1.2.3	$1-\left(\frac{5}{15}\times\frac{7}{14}+\frac{7}{15}\times\frac{5}{14}+\frac{7}{15}\times\frac{6}{14}+\frac{5}{15}\times\frac{4}{14}\right)$ $\frac{7}{15}$	✓method ✓computation ✓✓answer
Qn 2		**[8]**
2.1	$a=150$ $b=375$ $c=250$ $d=225$ $e=1075$	✓✓✓✓✓✓ 1 mark For each correct value
2.2	No ; Reason: Any logical explanation	✓No ✓reason

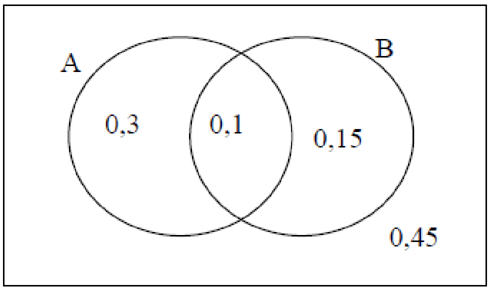
2.3	$P(m)\times P(w)=\frac{700}{1075}\times\frac{600}{1075}=\frac{672}{1849}=0{,}36$ $P(m\cap w)=\frac{450}{1075}=0{,}42$ $P(m)\times P(w)\neq P(m\cap w)$ $\therefore$ *NOT INDEPENDENT*	✓0,36 ✓0,42 ✓$P(m)\times$ $P(w)\neq$ $P(m\cap w)$ ✓conclusion
2.4.1	$\frac{75}{1075}=\frac{3}{43}=0{,}07$	✓answer
2.4.2	$\frac{600}{1075}+\frac{250}{1075}=\frac{850}{1075}=\frac{34}{43}=0{,}79$	✓+ ✓answer
2.4.3	$\frac{375}{1075}+\frac{250}{1075}-\frac{150}{1075}=\frac{19}{43}=0{,}44$	✓ substitution ✓answer
		[16]

PAST EXAMINATION QUESTIONS

Nov 2019 Solutions

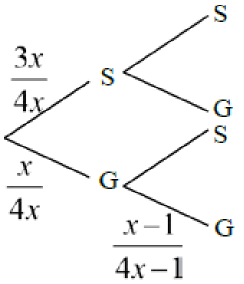
10.1	P(same day) $=\frac{4}{16}$ or $\frac{1}{4}$ or 0,25 or 25%	✓4 numerator ✓16 denominator (2)
10.2	P(2 consecutive days) $=\frac{3\times 2}{16}=\frac{3}{8}$	✓3 ✓×2 ✓ answer (3)
		[5]

Solutions

11.1.1	$P(A)\times P(B)$ independent events $=0{,}40\times 0{,}25 \;=0{,}1$ 	✓0,1 ✓0,15 and 0,3 ✓0,45 (3)
11.1.2	$P(\text{A or not B}) = P(\text{A}) + P(\text{not B}) - P(\text{A and not B})$ $=0{,}4+0{,}75-0{,}3$ $=0{,}85$	✓ substitution ✓answer (2)
	OR/OF $P(\text{A or not B}) = 1 - P(\text{only B})$ $= 1 - 0{,}15$ $= 0{,}85$ **OR/OF** From Venn diagram: $0{,}3 + 0{,}1 + 0{,}45 = 0{,}85$	**OR/OF** ✓ $1 - 0{,}15$ ✓answer (2) **OR/OF** ✓ substitution ✓ answer (2)
11.2	$(5\times 1\times 5) + (5\times 1\times 6) + (5\times 1\times 6) + (5\times 1\times 5) = 110$ $110\times 5 = 550 > 500$ Not possible, because not enough space	✓$5\times 1\times 5$ ✓$5\times 1\times 6$ ✓$5\times 1\times 6$ ✓$5\times 1\times 5$ ✓110 ✓ conclusion (6)

[11]

Nov 2018 Solutions

12.1	P(A or B) = P(A) + P(B) $0{,}74 = 0{,}45 + y$ $y = 0{,}29$	✓ P(A or B) = P(A) + P(B) ✓substitution ✓answer (3)
12.2	 Let the number of mystery gift bags = x The total number of bags = $4x$ $\left(\frac{x}{4x}\right)\times\left(\frac{x-1}{4x-1}\right)=\frac{7}{118}$ $\frac{1}{4}\times\frac{x-1}{4x-1}=\frac{7}{118}$	✓ $4x$ ✓$\left(\frac{x}{4x}\right)$ or $\left(\frac{1}{4}\right)$ ✓$\left(\frac{x-1}{4x-1}\right)$
	$\frac{x-1}{4x-1}=\frac{28}{118}$ $118x - 118 = 112x - 28$ $x = 15$	✓$\frac{1}{4}\times\frac{x-1}{4x-1}$ ✓ equating to $\frac{7}{118}$ ✓answer (6)

	OR/OF P(gift and gift) = P(gift at first draw) × P(gift at second draw) $\frac{7}{118}=\frac{1}{4}\times$ P(gift at second draw) P(gift at second draw) = $\frac{7}{118}\div\frac{1}{4}$ $=\frac{14}{59}$ Therefore: P(gift at first draw) $=\frac{15}{60}$ And: 15 bags had mystery gifts inside	**OR/OF** ✓$\frac{1}{4}$ ✓$\frac{1}{4}\times$ P(gift at 2^{nd} draw) ✓ $\frac{7}{118}=\frac{1}{4}\times$ P(gift at 2^{nd} draw) ✓$\frac{14}{59}$ ✓ $\frac{15}{60}$ ✓answer (6)
		[9]

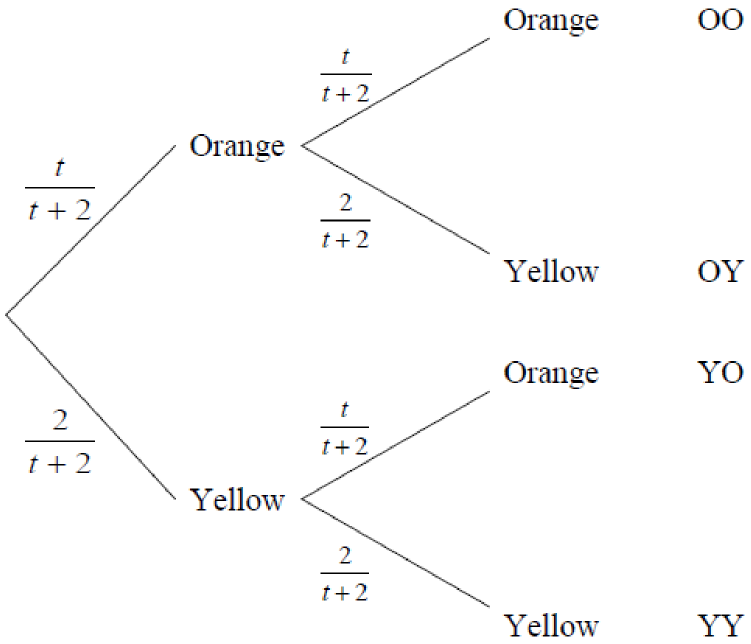
COUNTING PRINCIPLE

Nov 2018 Solutions

11.1.1	$7^5 = 16\ 807$	✓✓ answer (2)
11.1.2	$7\times6\times5\times4\times3$ $=\frac{7!}{2!}=2520$	✓ $7\times6\times5\times4\times3$ or $\frac{7!}{2!}$ ✓answer (2)
11.2	$2\times7\times1=14$	✓✓✓ $2\times7\times1$ (3)
		[7]

Nov 2015 Solutions

11.1	$P(A)\times P(B)$ $=0{,}2\times0{,}63$ $=0{,}126$ i.e. P(A) × P(B) = P(A and B) Therefore A and B are independent/*Dus is A en B onafhanklik*	✓$0{,}2\times0{,}63$ ✓$P(A)\times P(B)$=P(A and B) ✓conclusion (3)
11.2.1	$7^7 = 823\ 543$	✓✓7^7 (2)
11.2.2	7!=5040	✓✓7! (2)
11.2.3	There are 3 vowels $\Rightarrow$ 3 options for first position There are 4 consonants $\Rightarrow$ 4 options for last position The remaining 5 letters can be arranged in $5\times4\times3\times2\times1$ ways $3\times(5\times4\times3\times2\times1)\times4=1440$ *Daar is 3 klinkers $\Rightarrow$ 3 opsies vir die eerste posisie* *Daar is 4 konsonante $\Rightarrow$ 4 opsies vir die laaste posisie* *Die oorblywende 5 letters kan as volg gerangskik word* $5\times4\times3\times2\times1$ ways/*maniere* $3\times(5\times4\times3\times2\times1)\times4=1440$	✓$\times3$ ✓$\times4$ ✓ $5\times4\times3\times2\times1$ ✓answer (4)

11.3	 $\text{P(Orange, Orange)} + \text{P(Yellow, Yellow)} = \frac{52}{100}$	✓$\text{P(O)} = \left(\frac{t}{t+2}\right)$ ✓$\text{P(Y)} = \left(\frac{2}{t+2}\right)$
	$\left(\frac{t}{t+2}\right)\left(\frac{t}{t+2}\right)+\left(\frac{2}{t+2}\right)\left(\frac{2}{t+2}\right)=\frac{52}{100}$ $\frac{t^2}{t^2+4t+4}+\frac{4}{t^2+4t+4}=\frac{13}{25}$ $25(t^2+4)=13(t^2+4t+4)$ $3t^2-13t+12=0$ $(3t-4)(t-3)=0$ $t=3$ There are 3 orange balls in the bag/*Daar is 3 oranje balle in die sak*	✓$\text{P(O,O)} = \left(\frac{t}{t+2}\right)^2$ ✓ $\text{P(Y,Y)} = \left(\frac{2}{t+2}\right)^2$ ✓ $\left(\frac{t}{t+2}\right)\left(\frac{t}{t+2}\right)+\left(\frac{2}{t+2}\right)\left(\frac{2}{t+2}\right)=\frac{52}{100}$ ✓ $t=3$ (no ca) (6) **[17]**
	TOTAL/TOTAAL:	**150 marks**

Prelim Gauteng 2016   Solutions

12.1	9! = 362 880 (Any other valid representation of the answer)	✓answer
		(1)
12.3	$\frac{9!}{5!}$ = 3 024 **OR** 9 x 8 x 7 x 6 = 3 024	✓ 9! ✓ 5! ✓answer ✓✓ 9 x 8 x 7 x 6 ✓answer
		(3)
		[4]

EXEMPLAR PAPER

PAPER ONE
GRADE 12

MATHEMATICS
EXEMPLAR

MARKS: 150

TIME: 3 Hours

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions.

1. This question paper consists of 10 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.
6. If necessary, round off answers to TWO decimal places, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. An information sheet with formulae is included at the end of this question paper.
9. Number the answers correctly according to the numbering system used in this question paper.
10. Write neatly and legibly.

QUESTION 1

1.1 Solve for x:

1.1.1 $2x^2 - 9x - 5 = 0$ (3)

1.1.2 $x - \frac{2}{x} = -5$ (correct to TWO decimal places) (5)

1.1.3 $3x^2 - 16x \leq -5$ (4)

1.1.4 $x + 5 = \sqrt{3 - 3x}$ (4)

1.2 Given: $2(x-p)^2 - 17 = 2x^2 - qx + 1$

Find the value(s) of p and q. (5)

1.3 Given $P(t) = kt^2 + 4t - 5$.

Determine the value of k for which $P(t)$ has real roots. (3)

[24]

QUESTION 2

2.1 Consider the quadratic sequence: $4\,;3\,;\,-2;\,-11\,;\,-24\,;\ldots$

2.1.1 Write down the values of the next TWO terms of the sequence. (2)

2.1.2 Determine the general term of the sequence in the form $T_n = an^2 + bn + c$. (4)

2.1.3 Which term of the sequence has a value just after $-19\,499$. (4)

2.2 The sum of the first n terms of a series is given by the formula $S_n = 3^{n+1} - 3$.

2.2.1 Determine the sum of the first 5 terms. (1)

2.2.2 Determine the first 3 terms of the sequence. (4)

2.3 Evaluate:

$$\sum_{n=4}^{7} 5 \cdot 2^{-n}$$

(4)

[19]

QUESTION 3

Consider the sequence: $3\,;9\,;27;\ldots$

Aslam says that the fourth term of the sequence is 81.

Zinhle disagrees and says that the fourth term of the sequence is 57.

3.1 Explain why Aslam and Zinlhe could both be correct. (2)

3.2 Aslam and Zinhle continue with their number patterns.

Determine a formula for the n^{th} term of

3.2.1 Aslam's sequence. (1)

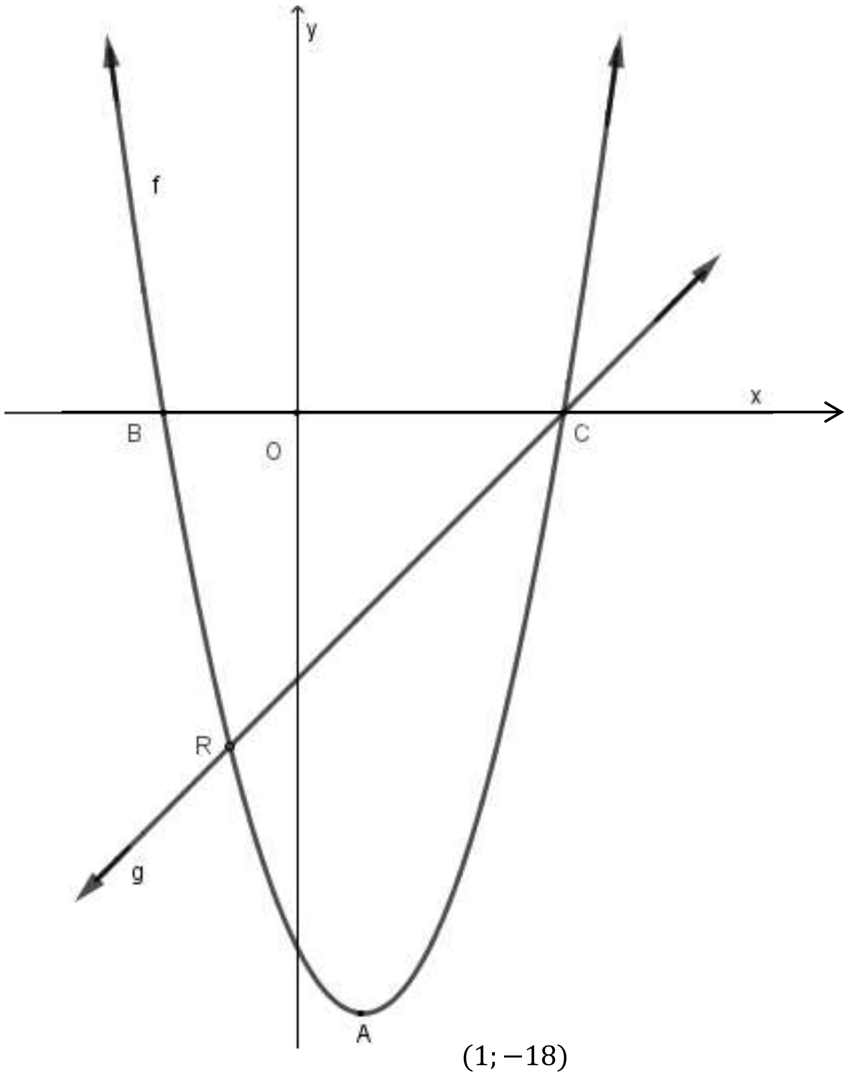
3.2.2 Zinhle's sequence. (4)

[7]

QUESTION 4

$A(1;-18)$ is the turning point of the graph of $f(x) = ax^2 + bx + c$. B and C are x –intercepts of f.

The graph of $g(x) = 2x - 8$ has an x –intercept at C. R is a point of intersection of f and g.



4.1 Calculate the coordinates of C. (2)

4.2 Determine the equation of f in the form $f(x) = ax^2 + bx + c$.

Show ALL your working. (4)

4.3 If $f(x) = 2x^2 - 4x - 16$, calculate the coordinates of R. (4)

4.4 Use your graphs to determine the value(s) of x for which

4.4.1 $f(x) \geq g(x)$. (2)

4.4.2 $f(x)$ is strictly increasing. (2)

4.5 Given that $h(x) = 2x^2 - 8x - 10$.

4.5.1 Describe the transformation of $f(x)$ to produce $h(x)$. (3)

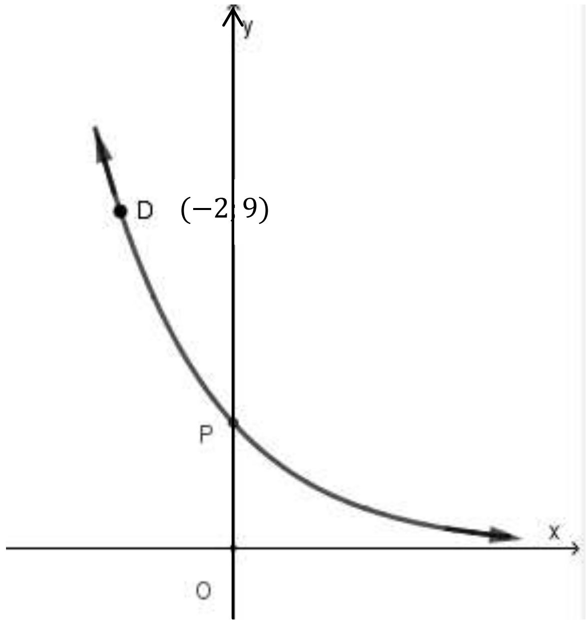
4.5.2 Write down the equation of the symmetry line of $h(x)$. (2)

[19]

QUESTION 5

The graph of $g(x) = b^x$ is drawn in the sketch below.

The point $D(-2; 9)$ lies on g. P is the y –intercept of g.



5.1 Write down the coordinates of P. (2)

5.2 Calculate the value of b. (2)

5.3 The graph h is obtained by reflecting g in the x –axis and shifting 2 units downwards. Write down the equation of h. (3)

5.4 Determine g^{-1} the inverse of g. (2)

5.5 Draw a neat sketch graph of g^{-1}. Clearly show all intercepts with the axes and any other point on the graph. (3)

5.6 For which value(s) of x is $g^{-1}(x) \geq 0$. (3)

[15]

QUESTION 6

6.1 Convert an interest rate of $10\%\ p.a$ compounded monthly to an interest rate per annum compounded quarterly. (3)

6.2 Sarah bought a bakkie in 2019 for her small furniture delivery business. The bakkiecosts$R85000$. The car dealership manager told her that it will depreciate in value at a rate of $20\%\ p.a$.

6.2.1 Calculate the cost price of Sarah's bakkie in 2024 using the reducing balance method. (3)

6.2.2 Sarah wishes to replace her old bakkie with a new one in 2024 when she sells the old bakkie. The proceeds from the sell of the old bakkie will be used together with some additional amount to pay for the new bakkie. If the price of a new bakkie increases at a rate of $5\%\ p.a$. Sarah decided to deposit some money every month in a fund earning interest of $7\%\ p.a$ compounded monthly. How much does she deposit monthly into the fund in order to be able to pay for the new bakkie in 2024.? Assume that her first deposit was made one month after buying the old bakkie. (7)

[13]

QUESTION 7

7.1 Determine $f'(x)$ from first principles if it is given that $f(x) = \frac{2}{x} - 3$. (5)

7.2 Determine $\frac{dy}{dx}$ if ;

7.2.1 $y = \frac{3}{\sqrt[5]{x^2}} - \pi x + 1$ (3)

7.2.2 $y = 4a$ and $a = x^3 + 2x$ (3)

7.3 Determine the equation of the tangent to $f(x) = (x+2)^2 + 1$ at a point where $x = -5$. (6)

[17]

QUESTION 8

Given: $f(x) = ax^3 + bx^2 + cx + d$.

$f(-1) = f(2) = f(7) = 0$ $\quad\quad$ $f(5) = 36$ $\quad\quad$ $f'(5) = 0$

8.1 Show that $a = -1; b = 8 \; ; c = -5$ and $d = -14$. (6)

8.2 Determine the coordinates of the turning point of f. (5)

8.3 Sketch the graph of f, clearly indicating the intercepts with the axes and the turning point(s). (3)

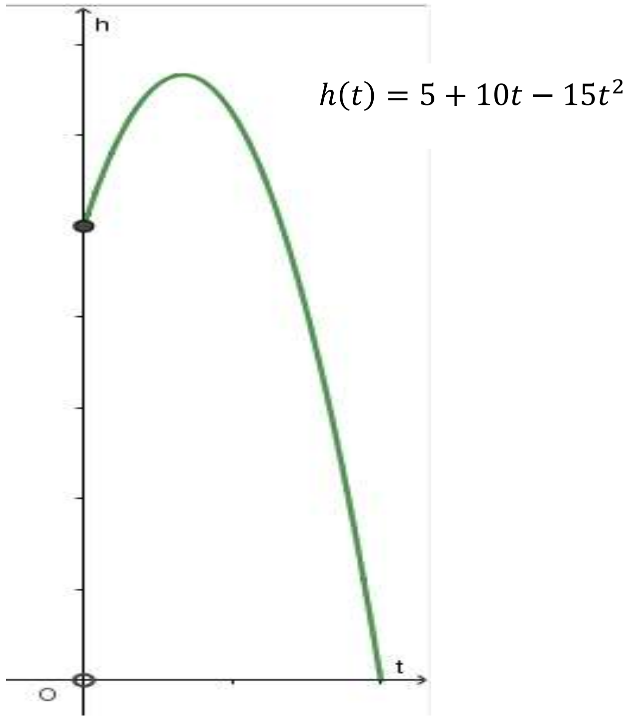
8.4 For which values of x will the graph of f be concave up? (3)

[17]

QUESTION 9

A fly is walking on the wall with a trajectory given by the formula $h(t) = 5 + 10t - 15t^2$, where h is the height measured in metres and t is time in seconds.

The motion of the fly is represented by the sketch below.



9.1 What is the height of the fly after $t = 0{,}7s$. (2)

9.2 How many seconds does it take for the fly to reach it's maximum height on the wall? (3)

[5]

QUESTION 10

10.1 In order to determine whether people aged between **20** and **25** years were employed or not, a survey was undertaken.
The tablebelow summarizes the results.

	MALES	FEMALES	TOTAL
Employed	30	130	160
Unemployed	140	100	240
TOTAL	170	230	400

Suppose that one of the people surveyed was chosen at random.
Determine the probability that the person:

10.1.1 is a male who is employed. (1)

10.1.2 is unemployed. (1)

10.1.3 is female, given that they are employed. (2)

10.1.4 Showing all working, determine whether it can be said that being employed is independent of gender. (3)

10.2 Three boys and four girls sit in a row watching a movie.
Determine the number of ways in which this can be done if:

10.2.1 they sit in any order. (1)

10.2.2 a girl sits at each end. (3)

10.2.3 the three boys do *not* sit together. (3)

[14]

TOTAL: 150

INFORMATION SHEET: MATHEMATICS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$A = P(1+ni)$ $\quad$ $A = P(1-ni)$ $\quad$ $A = P(1-i)^n$ $\quad$ $A = P(1+i)^n$

$\sum_{i=1}^{n} 1 = n$ $\quad$ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad$ $T_n = a + (n-1)d$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$T_n = ar^{n-1}$ $\quad$ $S_n = \frac{a\left(r^n - 1\right)}{r-1}$; $r \neq 1$ $\quad$ $S_\infty = \frac{a}{1-r}$; $-1 < r < 1$

$F = \frac{x\left[(1+i)^n - 1\right]}{i}$ $\quad$ $P = \frac{x[1-(1+i)^{-n}]}{i}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\quad$ $\mathrm{M}\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$y = mx + c$ $\quad$ $y - y_1 = m(x - x_1)$ $\quad$ $m = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad$ $m = \tan\theta$

$$(x-a)^2 + (y-b)^2 = r^2$$

In ΔABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $\quad$ $a^2 = b^2 + c^2 - 2bc.\cos A$ $\quad$ $area\,\Delta ABC = \frac{1}{2}ab.\sin C$

$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$ $\quad$ $\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$ $\quad$ $\cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}$$

$$\sin 2\alpha = 2\sin\alpha.\cos\alpha$$

$\bar{x} = \frac{\sum fx}{n}$ $\quad$ $\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$

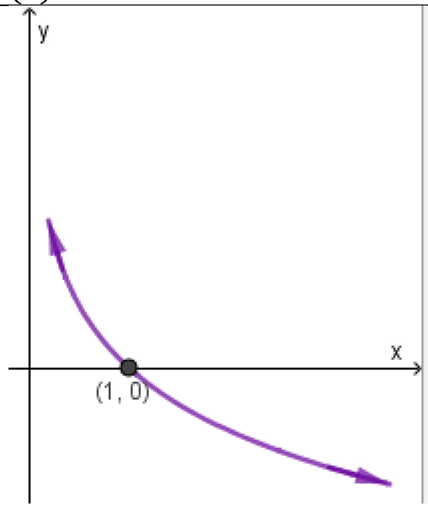
$P(A) = \frac{n(A)}{n(S)}$ $\quad$ $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$\hat{y} = a + bx$ $\quad$ $b = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$

EXEMPLAR PAPER MEMORANDUM	
QUESTION 1	
1.1 1.1.1	$2x^2 - 9x - 5 = 0$ $(2x+1)(x-5) = 0\checkmark$ $x = -\frac{1}{2}\checkmark\ OR\ \ x = 5\checkmark$ (3)
1.1.2	$x - \dfrac{2}{x} = -5$ $x^2 + 5x - 2 = 0\checkmark$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\checkmark$ $x = \dfrac{-(5) \pm \sqrt{(5)^2 - 4(1)(-2)}}{2(1)}\checkmark$ $x = 0{,}37\checkmark\ \ OR\ \ x = -5{,}37\checkmark$ (4)
1.1.3	$3x^2 - 16x \leq -5$ $3x^2 - 16x + 5 \leq 0\checkmark$ $(3x-1)(x-5) \leq 0\checkmark$ $C.V\ \dfrac{1}{3}\ \ and\ 5$ $x \leq \dfrac{1}{3}\checkmark\ \ OR\ x \geq 5\checkmark$ (4)
1.1.4	$x + 5 = \sqrt{3 - 3x}$ $x^2 + 10x + 25 = 3 - 3x\checkmark$ $x^2 + 13x + 22 = 0\checkmark$ $(x+2)(x+11) = 0$ $x = -2\ \ OR\ \ x = -11\ \ BOTH\ VALUES\ \checkmark$ $\therefore y = -2$ ONLY $\checkmark$ (4)
1.2	$2(x-p)^2 - 17 = 2x^2 - qx + 1$ $2(x^2 - 2xp + p^2) - 17 = 2x^2 - qx + 1$ $2x^2 - 4xp + 2p^2 - 17 = 2x^2 - qx + 1\ \checkmark$ $4p = q\checkmark$ $2p^2 - 17 = 1$ $p^2 = 9\checkmark$ $p = \pm 3\checkmark$ $q = \pm 12\checkmark$ (5)
1.3	$p(t) = kt^2 + 4t - 5$ $b^2 - 4ac \geq 0\checkmark$ $(4)^2 - 44(k)(-5) \geq 0\checkmark$ $k \geq -\dfrac{4}{5}\checkmark$ (3)
	[24]

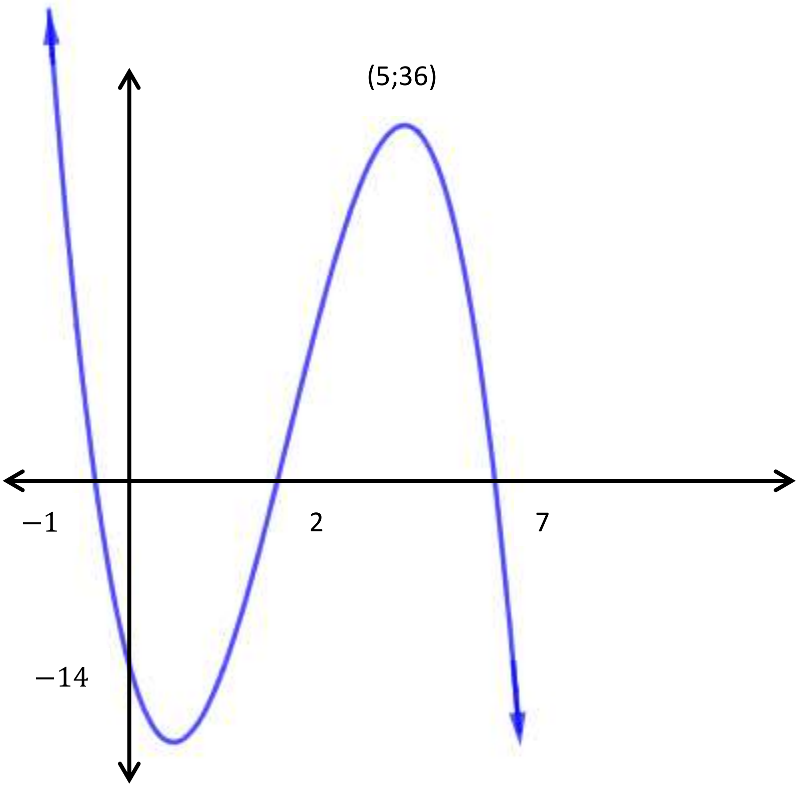
QUESTION 2	
2.1	$4\,;3\,;\,-2\,;\,-11\,;\,-24;\ldots$
2.1.1	$-41\checkmark\,;\,-62\checkmark$ (2)
2.1.2	$2a=-43a+b=-1a+b+c=4$ $a=-2\checkmark 3(-2)+b=-1-2+5+c=4$ $b=5\checkmark c=1\checkmark$ $T_n=-2n^2+5n+1\checkmark$ (4)
2.1.3	$T_n=-19\,499$ $-2n^2+5n+1=-14\,499\checkmark$ $-2n^2+5n+19\,500=0\checkmark$ $(2n+195)(n-100)=0$ $n=-97{,}5 \quad OR \quad n=100\ \ BOTH\ VALUES\ \checkmark$ $\therefore n=101^{th}\checkmark$ (4)
2.2 2.2.1	$S_n=3^{n+1}-3$ $S_5=3^{5+1}-3$ $S_5=726\checkmark$ (1)
2.2.2	$T_n=S_n-S_{n-1}\checkmark$ $T_1=S_1=6\checkmark$ $T_2=S_2-S_1T_3=S_3-S_2$ $T_2=24-6T_3=78-24$ $T_2=18\checkmark T_3=54\checkmark$ (4)
2.3	$\sum_{n=4}^{7}5.2^{-n}=5.2^{-4}+5.2^{-5}+5.2^{-6}+5.2^{-7}$ $=\frac{5}{16}+\frac{5}{32}+\frac{5}{64}+\frac{5}{128}\checkmark\checkmark$ $=\frac{75}{128}\checkmark$ (4) OR $a=\frac{5}{16}\quad r=\frac{1}{2}\checkmark$ $S_n=\frac{a(1-r^n)}{1-r}$ $S_4=\frac{\frac{5}{13}\left(1-\left(\frac{1}{2}\right)^4\right)}{1-\frac{1}{2}}\checkmark$ $=\frac{75}{128}\checkmark$
	[19]

QUESTION 3	
3.1	Aslam calculated that the sequence is geometric/ Exponential ✓ Zinhle calculated that the sequence is quadratic ✓ (2) OR Aslam has multiplied each term by 3 to get the next term Zinhle sees it as a sequence with a constant second difference.
3.2 3.2.1	$T_n = 3^n$✓ (1)
3.2.2	3 ; 9; 27; 57 6 18 30 12 12 $2a = 12 3a + b = 6 a + b + c = 3$ $a = 6$✓ $b = -12$✓ $c = 9$✓ $T_n = 6n^2 - 12n + 9$✓ (4)
	[7]
QUESTION 4	
4.1	$g(x) = 2x - 8$ $2x - 8 = 0$✓ $x = 4$ $C(4; 0)$✓ (2)
4.2	$f(x) = a(x - p)^2 + q$ $f(x) = a(x - 1)^2 - 18$ ✓ $0 = a(4 - 1)^2 - 18$ $a = 2$ ✓ $f(x) = 2(x - 1)^2 - 18$ $f(x) = 2(x^2 - 2x + 1) - 18$ ✓ $f(x) = 2x^2 - 4x - 16$ ✓ (4)
4.3	$2x^2 - 4x - 16 = 2x - 8$ ✓ $2x^2 - 6x - 8 = 0$ $(x - 4)(x + 1) = 0$ $x = 4 \quad OR \quad x = -1$ ✓ $y = 2(-1) - 8$✓ $y = -10$✓ $R(-1; -10)$✓ (4)
4.4 4.4.1	$x \leq -1$ ✓ $OR \quad x \geq 4$ ✓ (2)
4.4.2	$x > 1$ ✓✓ (2)

4.5 4.5.1	$h(x) = 2x^2 - 8x - 10$ $h(x) = 2[x^2 - 4x] - 10$ $h(x) = 2[(x-2)^2 - 2^2] - 10$ $h(x) = 2(x-2)^2 - 18$ ✓$f(x) = 2(x-1)^2 - 18$ $f(x)$is shifted 1 unit ✓ to the right ✓. (3)
4.5.2	$x = 2$✓✓ (2)
	[19]
QUESTION 5	
5.1	$y = b^0$✓ $y = 1$ $P(0;1)$✓ (2)
5.2	$9 = b^{-2}$✓ $b = \frac{1}{3}$✓ $g(x) = \left(\frac{1}{3}\right)^x$ ✓(3)
5.3	$g(x) = \left(\frac{1}{3}\right)^x$ $h(x) = -\left(\frac{1}{3}\right)^x - 2$ ✓✓(2)
5.4	$x = \left(\frac{1}{3}\right)^y$ ✓ $y = \log_{\frac{1}{3}} x$ $g^{-1}(\mathrm{x}) = \log_{\frac{1}{3}} x$ ✓ (2)
5.5	 ✓ shape ✓ asymptote ✓(1,0) (3)
5.6	$0 < x \leq 1$✓✓(2)
	[15]

QUESTION 6	
6.1	$\left(1+\frac{i_n}{n}\right)^n=\left(1+\frac{i_m}{m}\right)^m$ ✓ $\left(1+\frac{i_n}{4}\right)^4=\left(1+\frac{0{,}1}{12}\right)^{12}$ ✓ $i=0{,}100835648$ $i=10{,}08\%$ ✓ (3)
6.2 6.2.1	$A=P(1-i)^n$✓ $A=85000(1-0{,}2)^5$✓ $A=R27\,852{,}80$ ✓(3)
6.2.2	$A=P(1+i)^n$ $A=85000(1+0{,}05)^5$✓ $A=R108\,483{,}93$ ✓ $F=R108\,483{,}93-R27\,852{,}80$ $F=R80\,631{,}13$ ✓ $F=\frac{x[(1+i)^n-1]}{i}$✓ $80\,631{,}13=\frac{x\left[\left(1+\frac{0{,}07}{12}\right)^{60}-1\right]}{\frac{0{,}07}{12}}$✓60 ✓$\frac{0{,}07}{12}$ $x=R1\,126{,}24$ ✓(7)
	[13]
QUESTION 7	
7.1	$f(x)=\frac{2}{x}-3f(x+h)=\frac{2}{x+h}-3$ $f'(x)=\lim_{h=0}\frac{f(x+h)-f(x)}{h}$✓ $f'(x)=\lim_{h=0}\frac{\frac{2}{x+h}-3-\left(\frac{2}{x}-3\right)}{h}$✓ $f'(x)=\lim_{h=0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}$ $f'(x)=\lim_{h=0}\frac{\frac{-2h}{x(x+h)}}{h}$ $f'(x)=\lim_{h=0}\frac{-2h}{x(x+h)h}$✓ $f'(x)=\lim_{h=0}\frac{-2}{x(x+h)}$✓ $f'(x)=\frac{-2}{x(x+0)}$ $f'(x)=\frac{-2}{x^2}$✓ (5)

7.2 7.2.1	$y = 3x^{\frac{-2}{5}} \checkmark - \pi x + 1$ $\frac{dy}{dx} = -\frac{6}{5}x^{\frac{-2}{7}} \checkmark - \pi \checkmark$ $\frac{dy}{dx} = \frac{-6}{5x^{\frac{2}{7}}} - \pi$ (3)
7.2.2	$y = 4(x^3 + 2x)\checkmark$ $y = 4x^3 + 8x)$ $\frac{dy}{dx} = 12x^2 \checkmark + 8 \checkmark$ (3)
7.3	$y = (-5+2)^2 + 1 = 10\checkmark$ $(-5; 10)$ $f(x) = x^2 + 4x + 5 \quad \checkmark$ $f'(x) = 2x + 4\checkmark$ $f'(-5) = 2(-5) + 4$ $f'(x) = -6 \ \checkmark$ $y - y_1 = m(x - x_1)$ $y - 10 = -6(x+5)\checkmark$ $y = -6x - 20\checkmark$ (6)
	[17]
QUESTION 8	
8.1	$y = a(x - x_1)(x - x_2)(x - x_3)$ $y = a(x+1)(x-2)(x-7)\checkmark$ $36 = a(5+1)(5-2)(5-7)\checkmark$ $36 = -36a$ $a = -1\checkmark$ $y = -(x+1)(x-2)(x-7)$ $f(x) = -x^3 + 8x^2\checkmark - 5x\checkmark - 14\checkmark$ $b = 8 \quad ; c = -5 ; \quad d = -14$ (6)
8.2	$f'(x) = -3x^2 + 16x - 5\checkmark$ $-3x^2 + 16x - 5 = 0\checkmark$ $(3x-1)(x-5) = 0\checkmark$ $x = \frac{1}{3} \quad OR \quad x = 5 \ \checkmark both\ values$ $y = -14\frac{22}{27} = -14{,}81 \quad OR \quad y = 36$ ✓ both values $\left(\frac{1}{3} \ ; -14\frac{22}{27}\right) \quad and \quad (5; 36)(5)$

8.3	 ✓ intercepts ✓ turning points ✓ shape (3)
8.4	$f''(x) = -6x + 16$✓ $f''(x) > 0$ $-6x + 16 > 0$ ✓ $x < \frac{8}{3}$ $x < 2{,}67$✓ (3)
	[17]
QUESTION 9	
9.1	$h(t) = 5 + 10t - 15t^2$ $h(0{,}7) = 5 + 10(0{,}7) - 15(0{,}7)^2$✓ $= 4{,}65\ metres$ ✓ (2)
9.2	$h'(t) = 10 - 30t$ ✓ $10 - 30t = 0$ ✓ $t = \frac{1}{3}$✓ (3)
	[5]

QUESTION 10	
10.1 10.1.1	$\frac{30}{400}=\frac{3}{40}\checkmark$ (1)
10.1.2	$\frac{240}{400}=\frac{3}{5}\checkmark$ (1)
10.1.3	$\frac{130}{160}=\frac{13\checkmark}{16\checkmark}$ (2)
10.1.4	$P(A)\times P(B)=\frac{160}{400}\times\frac{170}{400}=\frac{17}{100}\checkmark$ $P(A\ and\ B)=\frac{30}{400}=\frac{3}{40}\checkmark$ $P(A)\times P(B)\neq P(A\ and\ B)\checkmark$ $\therefore$ *being employed is NOT indepedent of gender*$\checkmark$ (3)
10.2 10.2.1	$7!=5040\checkmark$ (1)
10.2.2	$4\times 3\checkmark\times 5!\checkmark=1440\checkmark$ (3)
10.2.3	Boys sitting together $5!\times 3!\checkmark=720\checkmark$ *OR* $5\times 3!\times 4!=720$ $7!-720=4320\checkmark$ (3)
	[14]
	TOTAL: 150

END