GRADE 12
Calculus 5 – Revise Factorising Cubic functions and Sketching Cubic Functions.
WEBSITE NOTES

TOPIC:

- Factorise third-degree polynomials
- Cubic graphs

Let us revise factorising of Cubic Functions

Example 1

If $f(x) = 2x^3+9x^2+3x-4$:

a. Show that x+4 is a factor of f(x).
b. Determine the remainder when f(x) is divided by x-2

Answer

a. To show that x+4 is a factor we need to substitute x = -4 (x+4=0 therefore x=-4) into the f(x). If the answer is 0 then it is a factor.
   $f(x) = 2x^3+9x^2+3x-4$
   $f(-4) = 2(-4)^3+9(-4)^2+3(-4)-4$
   $f(-4) = 0$
   Therefore x+4 is a factor

b. Substitute x = 2 (x-2 = 0 therefore x = 2) into f(x)
   $f(x) = 2x^3+9x^2+3x-4$
   $f(2) = 2(2)^3+9(2)^2+3(2)-4$
   $f(2) = 54$
   Therefore 54 is the remainder when f(x) is divided by x-2

Example 2 (Try yourself)

If $f(x) = x^3+3x^2-6x-8$:

a. Show that x-2 is a factor of f(x)
b. Determine the remainder when f(x) is divided by x-1

Answers:

a. $f(2) = (2)^3+3(2)^2-6(2)-8$
   f (2) = 0 Yes, it is a factor
b. $f(1) = (1)^3+3(1)^2-6(1)-8$
   f (1) = -10
   -10 is the remainder

Example 3 (Try yourself)

If $f(x) = 2x^3+x^2-ax+5$ find a if:

-2 is the remainder so if you substitute x = -1 into f(x) the answer will be 2

a. The remainder is 2 when f(x) is divided by x +1
   $f(-1) = 2(-1)^3+(-1)^2-a(-1)+5$
   $2 = -2+1+a+5$
   $2+2-1-5 = a$
   $-2 = a$

b. 2x-1 is a factor of f(x)
$f(x) = 2x^3+x^2-ax+5$
$0 = 2(½)^3+(½)^2-a(½)+5$
$0 = ¼ + ¼ - ½a + 5$
$a = 11$

If 2x-1 is a factor, then you will substitute

x = ½ into f(x)

WHY???

2x-1=0

2x = 1

x = ½

Example 4 (Try yourself)

Factorise $f(x) = x^3-2x^2-5x+6$

Answer

$f(x) = (x-1)(x-3)(x+2)$

Example 5 (Try yourself)

Factorise $f(x) = 2x^3+x^2-13x+6$

Answer

$f(x) = (x-2)(x+3)(2x-1)$

Example 6

Solve for x:

$x^3-2x^2-4x+8 = 0$

Answer

Find the first factor

$f(x) = x^3-2x^2-4x+8$

- x+2 =0 is a factor because f (-2) = 0

Divide f(x) by x+2 (x=-2) by using SYNTHETIC DIVISION (See previous notes)

- $f(x) = (x+2)(x^2-4x+4)$
- $f(x) = (x+2)(x-2)(x-2)$

Can also be factorised by using the QUADRATIC FORMULA

Equate f(x) = 0 and solve for x

- $0 = (x+2)(x-2)(x-2)$
- x+2 =0 or x-2 =0 or x-2 =0
- Therefore x = -2 or x = 2 or x = 2

NOTE THAT THESE X-VALUES REPRESENT THE ROOTS, X-INTERCEPTS OF THE FUNCTIONS. THIS IS WHERE THE X-AXIS IS CUT BY THE FUNCTION.

If x-a is a Factor of f(x), then x = a is a root of f(x) because f(a) = 0

Sketching Cubic Functions

To draw the Graph, you need to know

1. How to determine the Shape. (We will use the derivative to help us with that)
2. The y-intercept (when x = 0)
3. The x-intercepts (when y = 0 – we will use the synthetic division)
4. Stationary points – Turning points (using the derivative = 0)
5. Point of infliction (using the second derivative = 0)

Example 1

If $f(x) = x^3+3x^2-9x-27$ sketch the graph of f(x).

Answer

There are a few things that need to be worked out first before the graph is finally sketched. This type of question can be broken up into the different parts – by asking y-intercept, x-intercepts, point of inflection etc… separately.

1. **SHAPE**

Remember that f '(x) is the derivative and the derivative is the gradient of the function at point x. SO REMEMBER THAT f '(x) DETERMINES THE SLOPE OF THE FUNCTION.

If gradient is POSITIVE:	If gradient is NEGATIVE:	If gradient is 0:
/	\	—

Find the f '(x)
$f(x) = x^3+3x^2-9x-27$
$f'(x) = 3x^2+6x-9$

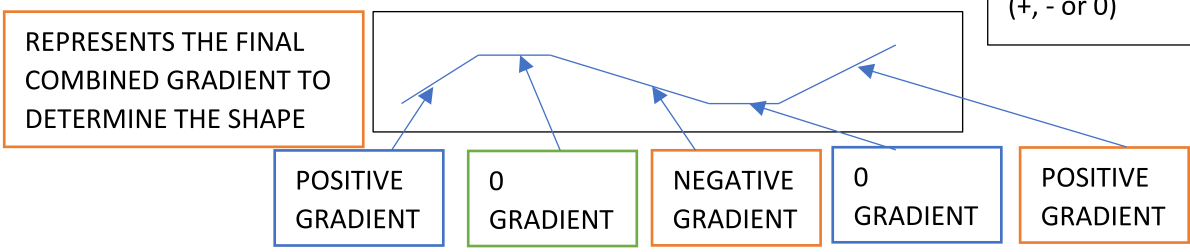
Factorise f '(x)
$f'(x) = 3x^2+6x-9$
$f'(x) = 3(x^2+2x-3)$
$f'(x) = 3(x+3)(x-1)$
Find the values of x if f(x) = 0

$(x+3)(x-1) = 0$
x = -3 or x = 1 THESE BECOME THE CRITICAL VALUES IN TABLE OF SIGNS.

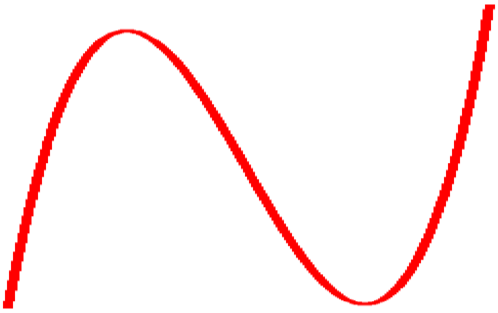
	ANY VALUE LESS THAN -3	**-3**	ANY VALUE BETWEEN -3 AND 1	**1**	ANY VALUE GREATER THAN 1
x+3	-	0	+	+	+
x-1	-	-	-	0	+
(x+3) (x -1)	+	0	-	0	+

With the table of signs we are interested with the sign and not the value.
(Whether it is + or – or 0 – this is going to help with the slope.)

The last row is the two expressions multiplied to get a combined sign (+, - or 0)



Shape will be



The points where the graph turns is where the gradient = 0

2. **<u>Y-INTERCEPT (WHERE X = 0)</u>**
   $f(x) = x^3+3x^2-9x-27$
   $f(0) = (0)^3+3(0)^2-9(0)-27$
   $f(0) = -27$
   (0; -27)

3. **<u>X-INTERCEPTS (WHERE Y = 0)</u>**

   $f(x) = x^3+3x^2-9x-27$
   $0 = x^3+3x^2-9x-27$

   Factorise and solve for x as per previous notes

   $(x-3)(x+3)(x+3) = 0$

   x = 3 or x = -3 or x = -3

   (3; 0) (-3; 0) (-3; 0) are the x-intercepts. There are two x-intercepts that are the same which means the function will turn at that point.

4. **<u>TURNING POINTS</u>**
   $f'(x) = 3(x+3)(x-1)$ (FROM POINT 1)
   x = -3 or x = 1
   SUBSTITUTE x = -3 and x = 1 into $f(x) = x^3+3x^2-9x-27$

   $f(-3) = (-3)^3+3(-3)^2-9(-3)-27$
   $f(-3) = 0$
   (-3; 0) is the one TURNING POINT

   $f(1) = (1)^3+3(1)^2-9(1)-27$
   $f(1) = -32$
   (1; -32) is the other TURNING POINT

5. **<u>POINT OF INFLICTION (WHERE THE CONCAVITY CHANGES)</u>**

   We make use of the SECOND DERIVATIVE
   (DERIVATIVE OF THE DERIVATIVE) – SEE PREVIOUS NOTES

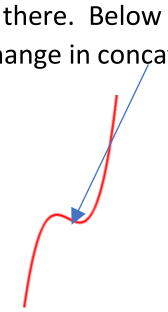
   $f'(x) = 3x^2+6x-9$
   $f''(x) = 6x + 6$
   $0 = 6x + 6$

<u>NOTE:</u>

<u>CONCAVE UP</u>

<u>CONCAVE DOWN</u>

Sometimes you can't see the concavity in a cubic graph, but it is still there. Below you can see the change in concavity



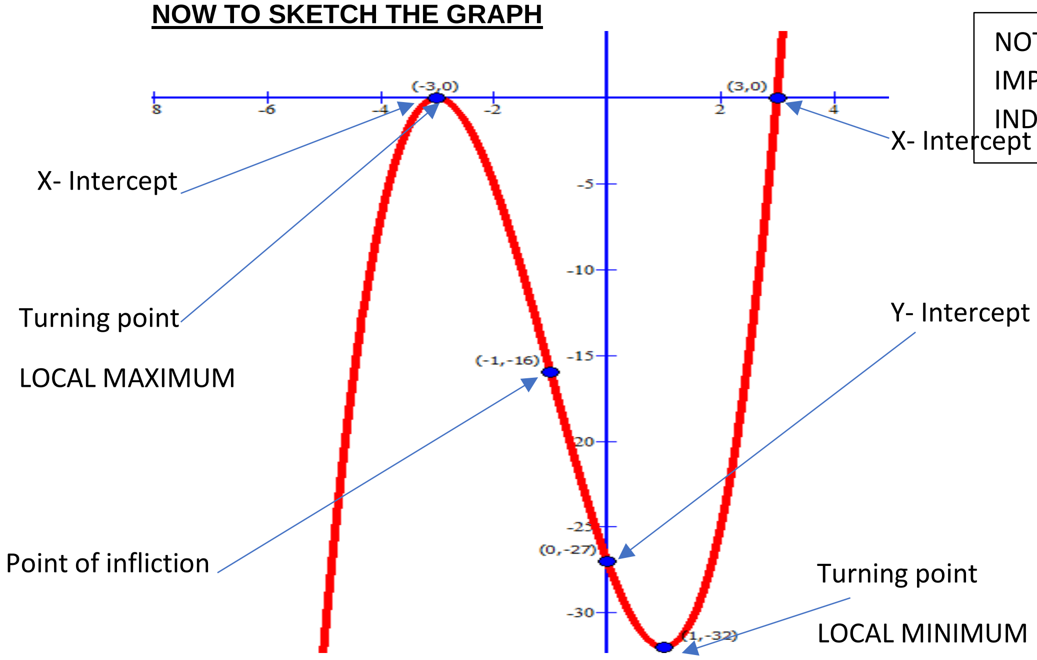
-6 = 6x
-1 = x
SUBSTITUTE X = -1 INTO THE ORIGINAL EQUATION
$f(x) = x^3+3x^2-9x-27$
$f(-1) = (-1)^3+3(-1)^2-9(-1)-27$
$f(-1) = -16$

(-1; -16) is the point of infliction

ALL THIS WORKING OUT ABOVE IS USED TO SKETCH THE GRAPH AND AS YOU PRACTICE MORE YOU DON'T HAVE TO DO IT IN SO MUCH DETAIL AS ABOVE – UNLESS A SPECIFIC QUESTION IS ASKED LIKE – WHAT IS THE POINT OF INFLICTION OR WHAT ARE THE TURNING POINTS.

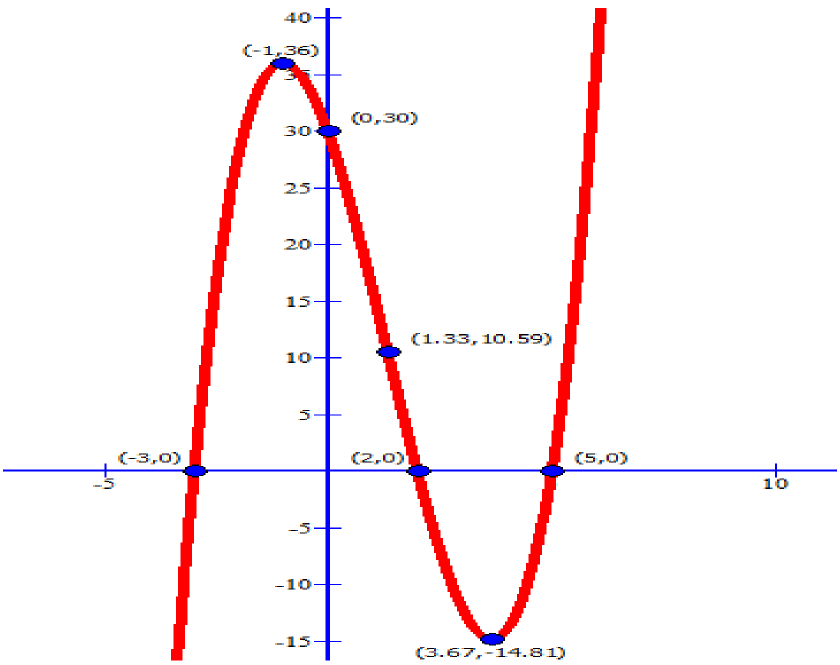
NOW TO SKETCH THE GRAPH

NOTE THAT ALL THE IMPORTANT POINTS ARE INDICATED ON GRAPH



Example 2 (Try yourself)
Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$. Indicate on the graph the following: y- intercept ; x-intercepts; Turning points, point of infliction

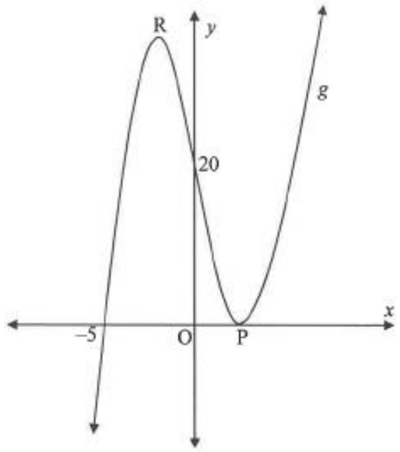
Answer



Example 3 (Try yourself) – PAST PAPER

QUESTION 9

9.1 The graph of $g(x) = x^3 + bx^2 + cx + d$ is sketched below.
The graph of g intersects the x-axis at $(-5\,;0)$ and at P, and the y-axis at $(0\,;20)$.
P and R are turning points of g.



This time you are given the graph and must answer questions pertaining to the graph. Knowledge of how to draw the graph is needed.

9.1.1	Show that $b = 1$, $c = -16$ and $d = 20$.	(4)
9.1.2	Calculate the coordinates of P and R.	(5)
9.1.3	Is the graph concave up or concave down at $(0\,;20)$? Show ALL your calculations.	(3)

HINT:

Where f''(x) = 0 is the point of infliction.

f''(x) > 0 – concave up

f''(x) < 0 – concave down

Answer

QUESTION/*VRAAG* 9

9.1.1	$g(x) = (x+5)(x-x_1)^2$ $20 = 5(x_1)^2$ $x_1^2 = 4$ $x_1 = 2$ $g(x) = (x+5)(x-2)^2$ $g(x) = (x+5)(x^2-4x+4)$ $g(x) = x^3 + x^2 - 16x + 20$	✓ $(x+5)$ ✓repeated root ✓ $x_1 = 2$ ✓ $g(x) = (x+5)(x^2-4x+4)$ (4)
9.1.2	$g(x) = x^3 + x^2 - 16x + 20$ $g'(x) = 3x^2 + 2x - 16$ $3x^2 + 2x - 16 = 0$ $(3x+8)(x-2) = 0$ $x = \frac{-8}{3}$ or $x = 2$ $R\left(\frac{-8}{3};\frac{1372}{27}\right)$ or $R(-2{,}67;50{,}81)$ $P(2;0)$	✓derivative ✓equating to zero ✓factors ✓co-ordinates of R ✓ co-ordinates of P (5)
9.1.3	$g''(x) = 6x + 2$ $g''(0) = 2$ $\therefore$ concave up	✓ $g''(x) = 6x + 2$ ✓ $g''(0) = 2$ ✓conclusion (3)