Grade 12
Calculus
Gradient at a Point and First Principles

First let us revise **Average Gradient** between two points.

Example 1

Find the average Gradient of function f(x) = $2x^2$ between the values x=2 and x=3.

Answer

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We know that y = f(x).

This means that $\mathbf{y_2=f(x_2)}$ and $\mathbf{y_1=f(x_1)}$ in this example.

Therefore, we can write the average gradient as follows:

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We must now obtain the y values by substituting the x values into the formula.

$$f(x) = 2x^2$$

SUBSTITUTE x= 3 and x=2 to obtain the y-values

USING x=3

$$f(3) = 2.(3)^2$$

$$f(3) = 18$$

USING x=2

$$f(2) = 2.(2)^2$$

$$f(2) = 8$$

Now we can substitute into the Average Gradient Formula

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{f(3) - f(2)}{3 - 2}$$

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{18 - 8}{3 - 2}$$

$$Average\ Gradient = \frac{\Delta y}{\Delta x} = \frac{10}{1}$$

$$\therefore Average\ Gradient\ of\ the\ function\ f(x) = 2x^2\ between\ x = 3\ and\ x = 2\ is\ 10$$

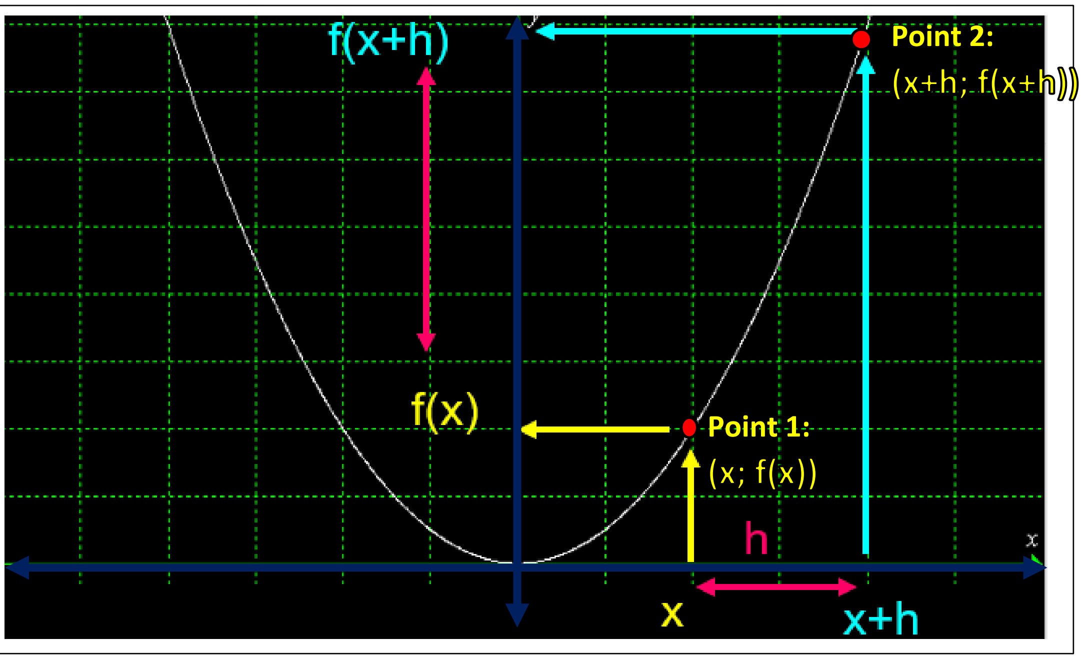
Now to look at **GRADIENT AT A POINT.**

- Consider the Average Gradient between two points on a curve.
- When we work with the GRADIENT AT A POINT, these **two points are brought closer and closer together**, until the **distance between the two points tends towards zero** (i.e. the two points "**merge**" into one point).
- If **h** represents distance between the two points, then as the **distance tends towards zero**, we write it as $\boldsymbol{h \to 0}$

- Considering the Average Gradient between two points. If we take let the **first x-coordinate** to be **x**, then the second x-coordinate is **h units away** (**distance between the two points**) and therefore the **second x-coordinate** would be ***x+h.***
- We can rewrite the average gradient formula as:

$$Average\ Gradient = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

- GRAPHICALLY TO EXPLAIN FORMULA



SIMPLIFYING THE AVERAGE GRADIENT FORMULA:

$$Average\ Gradient = \frac{f(x+h) - f(x)}{h}$$

If the **distance between the 2 values of x becomes smaller and smaller** so the **second value of x (at Point 2) gets closer and closer to the first value of x (Point 1)** and so the **distance between the two points, *h***, becomes very, very small and **approaches zero**. It is therefore approaching a specific point.

h tends towards 0 is written as $\lim\limits_{h \to 0}$

- Therefore, the Gradient at a point is written as

$$Gradient\ at\ a\ point\ or\ Istantaneous\ Gradient = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- This is the **DERIVATIVE** of a function (**indicated by f '(x)**) and is known as **THE FIRST PRINCIPLES FORMULA.**

THEREFORE $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Example 2

Find the Derivative of the function f(x) = 2x² using FIRST PRINCIPLES

Answer

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is the formula to be used.

- **USING x substituted into the f(x)**

  $f(x) = 2x^2$

- **USING (x+h) substituted into the f(x)**

  $f(x+h) = 2(x+h)^2$
  $f(x+h) = 2(x+h).(x+h)$
  $f(x+h) = 2(x^2 + 2xh + h^2)$
  $f(x+h) = 2x^2 + 4xh + 2h^2$

- **SUBSTITUTE INTO FIRST PRINCIPLES FORMULA TO CALCULATE f '(x)**

  $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

  $f'(x) = \lim_{h \to 0} \frac{(2x^2+4xh+2h^2)-(2x^2)}{h}$

- **SIMPLIFY**

  $f'(x) = \lim_{h \to 0} \frac{2x^2+4xh+2h^2-2x^2}{h}$

  $f'(x) = \lim_{h \to 0} \frac{4xh+2h^2}{h}$

- **TAKE OUT COMMON FACTOR**

  $f'(x) = \lim_{h \to 0} \frac{2h(2x+h)}{h}$

- **CANCEL DENOMINATOR WITH NUMERATOR**

  $f'(x) = \lim_{h \to 0} \frac{2\cancel{h}(2x+h)}{\cancel{h}}$

  $f'(x) = \lim_{h \to 0} 2x + h$

- **NOW SUBSTITUTE h=0 INTO THE FORMULA**

  $f'(x) = 2x + (0)$

YOU CANNOT SUBSTITUTE h = 0 WHILE THE DENOMINATOR h IS IN EQUATION. THIS WILL RESULT IN DIVIDING BY 0 WHICH WILL BE UNDEFINED.

YOU NEED TO SIMPLIFY, FACTORISE AND THEN CANCEL.

$\lim_{h \to 0}$ is removed because we are substituting h=0 into the formula now.

$\therefore f'(x) = 2x$

THIS MEANS THE FOLLOWING –

THE DERIVATIVE (OR GRADIENT) IS EQUAL TO $2x$ AT THE POINT x.

NB!!!! WHEN ASKED TO SOLVE USING FIRST PRINCIPLES THE ABOVE STEPS MUST BE USED.

LET US LOOK AT ANOTHER EXAMPLE ON HOW YOU WOULD ANSWER IN AN EXAM OR TEST.

Example 3

Find the Derivative of the function f(x) = x^2-5 using **FIRST PRINCIPLES**

Answer

$\boldsymbol{f(x) = x^2 - 5}$

$f(x+h) = (x+h)^2 - 5$

$f(x+h) = (x^2 + 2xh + h^2) - 5$

$\boldsymbol{f(x+h) = x^2 + 2xh + h^2 - 5}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x^2+2xh+h^2-5) - (x^2-5)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x^2+2xh+h^2-5 - x^2+5}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2h(x+h)}{h}$$

$$f'(x) = \lim_{h\to 0} 2(x+h)$$

$$f'(x) = 2(x + (0))$$

$$f'(x) = 2x$$

NOTE:
The negative sign changes to a positive sign when the brackets are dropped because you are multiplying the negative into the bracket
$\boldsymbol{-(x^2 - 5) = -x^2 + 5}$

Exercise 1

Find the $\boldsymbol{f'(x)}$ of the following functions using **FIRST PRINCIPLES:**

a. $f(x) = x^2 - 5$

b. $f(x) = x^2 + 2$

c. $f(x) = 4 - 7x$