GRADE 11

Functions 6

Average gradient and gradient of a curve at a point and Practical problems.

WEBSITE NOTES ANSWERS

TOPIC:

- Average gradient and gradient of a curve at a point.
- Practical problems

Average Gradient

Example 1

Consider f(x) = x^2+5x+6

Determine the average gradient between:

1. x = -1 and x = 3
2. x = -1 and x = 2
3. x = -1 and x = 1
4. x = -1 and x = 0

To obtain the y values for the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the given x-values into the function.

Answer

1. x_2 = -1 to get y_2

   f (-1) = $(-1)^2$+5. (-1) +6
   f (-1) = 1-5+6
   f (-1) = 2

   x_1 = 3 to get y_1
   f (3) = $(3)^2$+5. (3) +6
   f (3) = 9 +15 +6
   f (3) = 30

   $m = \frac{y_2 - y_1}{x_2 - x_1}$

   $m = \frac{2-30}{-1-3} = \frac{-28}{-4} = 7$

   $Average\ Gradient\ is\ 7\ between\ x = -1\ and\ x = 3$

2. x_2 = -1 to get y_2

   f (-1) = $(-1)^2$+5. (-1) +6
   f (-1) = 1-5+6
   f (-1) = 2

   x_1 = 2 to get y_1
   f (2) = $(2)^2$+5. (2) +6
   f (2) = 4 + 10 +6
   f (2) = 20

   $m = \frac{y_2 - y_1}{x_2 - x_1}$

   $m = \frac{2-20}{-1-2} = \frac{-18}{-3} = 6$

   $Average\ Gradient\ is\ 6\ between\ x = -1\ and\ x = 2$

3. x_2 = -1 to get y_2
   f (-1) = $(-1)^2$+5. (-1) +6
   f (-1) = 1-5+6
   f (-1) = 2

   x_1 = 1 to get y_1
   f (1) = $(1)^2$+5. (1) +6
   f (1) = 1+5+6
   f (1) = 12

   $m = \frac{y_2 - y_1}{x_2 - x_1}$
   $m = \frac{2-12}{-1-1} = \frac{-10}{-2} = 5$
   $Average\ Gradient\ is\ 5\ between\ x = -1\ and\ x = 1$

4. x_2 = -1 to get y_2
   f (-1) = $(-1)^2$+5. (-1) +6
   f (-1) = 1-5+6
   f (-1) = 2

   x_1 = 0 to get y_1
   f (0) = $(0)^2$+5. (0) +6
   f (0) = 6

   $m = \frac{y_2 - y_1}{x_2 - x_1}$
   $m = \frac{2-6}{-1-0} = \frac{-4}{-1} = 4$
   $Average\ Gradient\ is\ 4\ between\ x = -1\ and\ x = 0$

If we look at the above example, we notice:

- As the distance between the two points reduces the gradient decreases.
- We used x = -1 for the one point in all 4 questions as instructed.

We can therefore conclude the following:

- As the distance between the two x points decrease, we are moving to a single point – the gradient at a point (x = - 1 because this point doesn't change) and not just average gradient.
- The Average Gradient that will best describe the Gradient at the point x = -1 would be the Average Gradient that has the smallest distance between the x-values. This would be 4 in the above example because the distance between the two x-values differs by only 1.
- NOTE that this is not the average gradient at the point x=-1 only the Gradient that is the closest to the gradient at a point.

$\frac{f(x+h)-f(x)}{(x+h)-(x)}$

NOTE:

- is another way of writing the gradient.
- is important to remember for Grade 12
- h is the distance between two points. (if $x_2 = 3$ and $x_1 = 4$ then h will be $x_2 - x_1 = 3 - 4 = -1$)
- So, the smaller h becomes the closer to the gradient at a point the average gradient becomes.

<u>Gradient at a point</u>

<u>Example 2</u>

Consider f(x) = x^2+5x+6

Determine the Gradient at the point (-1; 2)

Answer

This is done before we work with the point given

$$m = \frac{f(x+h)-f(x)}{(x+h)-(x)}$$

- Write down f(x)

  $f(x) = x^2 + 5x + 6$
- Determine f(x+h)

  $f(x+h) = (x+h)^2 + 5(x+h) + 6$

  $f(x+h) = (x^2 + 2xh + h^2) + (5x + 5h) + 6$

  $f(x+h) = x^2 + 2xh + h^2 + 5x + 5h + 6$
- Determine f(x+h) – f(x)

  $f(x+h) - f(x) = (x^2 + 2xh + h^2 + 5x + 5h + 6) - (x^2 + 5x + 6)$

  $f(x+h) - f(x) = x^2 + 2xh + h^2 + 5x + 5h + 6 - x^2 - 5x - 6$

  $f(x+h) - f(x) = 2xh + h^2 + 5h$
- Determine $\frac{f(x+h)-f(x)}{(x+h)-(x)}$
    - $= \frac{2xh+h^2+5h}{x+h-x}$
  - $\boldsymbol{x + h - x = h}$
    - $= \frac{2xh+h^2+5h}{h}$
  - **<u>Factorise the numerator by taking out common factor h</u>**
    - $= \frac{h(2x+h+5)}{h}$
  - **<u>Simplify the expression by cancelling the h in the numerator with h in the denominator.</u>**
    - $= 2x + h + 5$
  - $\boldsymbol{2x + 5}$ is the gradient at a point where h=0 because there would be no difference between two points because there is only one point now.

To work out the gradient at the point (-1;2) you would substitute x=-1 into 2x+5. The y=2 we don't need to use.

2(-1) + 5 = 3

Therefore 3 is the gradient of f(x) = x^2+5x+6 at the point (-1; 2)

<u>Example 3 (Try yourself)</u>

Consider f(x) = x^2+6x+8

1. Determine the average gradient between:
   a. x = 1 and x = 4
   b. x = 1 and x = 3
   c. x = 1 and x = 2
2. Which of the above average gradients would best describe the gradient of f(x) at the point x = 1?
3. Determine the gradient of f(x) = x^2+6x+8 at the point (1; 15)

Answers

1.
   a. x_2 = 1 to get y_2
      f (1) = $(1)^2$+6. (1) +6
      f (1) = 1+6+6
      f (1) = 13

      x_1 = 4 to get y_1
      f (4) = $(4)^2$+6. (4) +6
      f (4) = 16 +24 +6
      f (4) = 46

      $m = \frac{y_2 - y_1}{x_2 - x_1}$
      $m = \frac{13-46}{1-4} = \frac{-33}{-3} = 11$
      $Average\ Gradient\ is\ 11\ between\ x = 1\ and\ x = 4$

   b. x_2 = 1 to get y_2
      f (1) = $(1)^2$+6. (1) +6
      f (1) = 1+6+6
      f (1) = 13

      x_1 = 3 to get y_1
      f (3) = $(3)^2$+6. (3) +6
      f (3) = 9+18+6
      f (3) = 33

      $m = \frac{y_2 - y_1}{x_2 - x_1}$
      $m = \frac{13-33}{1-3} = \frac{-20}{-2} = 10$
      $Average\ Gradient\ is\ 10\ between\ x = 1\ and\ x = 3$

   c. x_2 = 1 to get y_2
      f (1) = $(1)^2$+6. (1) +6
      f (1) = 1+6+6
      f (1) = 13

      x_1 = 2 to get y_1
      f (2) = $(2)^2$+6. (2) +6
      f (2) = 4 +12 +6
      f (2) = 22

      $m = \frac{y_2 - y_1}{x_2 - x_1}$
      $m = \frac{13-22}{1-2} = \frac{-9}{-1} = 9$
      $Average\ Gradient\ is\ 9\ between\ x = 1\ and\ x = 2$

2. 9
3. $m = \frac{f(x+h) - f(x)}{(x+h) - (x)}$
   - Write down f(x)

$f(x) = x^2 + 6x + 8$

- Determine f(x+h)

  $f(x+h) = (x+h)^2 + 6(x+h) + 8$

  $f(x+h) = (x^2 + 2xh + h^2) + (6x + 6h) + 8$

  $f(x+h) = x^2 + 2xh + h^2 + 6x + 6h + 8$

- Determine f(x+h) – f(x)

  $f(x+h) - f(x) = (x^2 + 2xh + h^2 + 6x + 6h + 8) - (x^2 + 6x + 8)$

  $f(x+h) - f(x) = x^2 + 2xh + h^2 + 6x + 6h + 8 - x^2 - 6x - 8$

  $f(x+h) - f(x) = 2xh + h^2 + 6h$

Determine $\frac{f(x+h)-f(x)}{(x+h)-(x)}$

$= \frac{2xh+h^2+6h}{x+h-x}$

<u>$\boldsymbol{x + h - x = h}$</u>

$= \frac{2xh+h^2+6h}{h}$

<u>Factorise the numerator by taking out common factor h</u>

$= \frac{h(2x+h+6)}{h}$

<u>Simplify the expression by cancelling the h in the numerator with h in the denominator.</u>

$= 2x + h + 6$

$\mathbf{2x + 6}$ is the gradient at a point where h=0 because there would be no difference between two points because there is only one point now.

To work out the gradient at the point (1;15) you would substitute x=1 into 2x+6. The y=15 we don't need to use.

2(1) + 6 = 8

Therefore 8 is the gradient of f(x) = x^2+6x+8 at the point (1; 15)

<u>Practical Problems on functions</u>

<u>Example 4</u>

Currently the subscription to a gym for a single member is R500 annually while family membership is R800. The gym is considering changing all memberships fees by the same amount. If this is done, then the single membership will cost $\frac{1}{2}$ of the family membership. Determine the proposed change.

<u>Answer</u>

Step 1: Summarise the information in a table

Let the proposed increase be x.

	Now	After
Single	R500	R500 + x
Family	R800	R800 + x

Step 2: Set up an equation

500+x = ½ (800+x)

Step 3: Solve the equation.

500+x = ½ (800+x)

1000 + 2x = 800 + x

1000 – 800 =-2x +x

200 = -x

x = -200

Step 4: Write down the answer

The change will be a reduction of R200

Example 5 (try yourself)

Currently the cost for a medical aid for a single member is R2000 annually while family membership is R5000. The medical aid is considering changing all memberships fees by the same amount. If this is done, then the single membership will cost $\frac{2}{3}$ of the family cost.

Determine the proposed change.

Answer

Step 1: Summarise the information in a table

Let the proposed increase be x.

	Now	After
Single	R2000	R2000 + x
Family	R5000	R5000 + x

Step 2: Set up an equation

$2000+x = \frac{2}{3}(5000+x)$

Step 3: Solve the equation.

$2000+x = \frac{2}{3}(5000+x)$

$6000 + 3x = 10000 + 2x$

$3x - 2x = 10000 - 6000$

$x = 4000$

Step 4: Write down the answer

The change will be an increase of R4000

Example 6 (try yourself)

When an object is dropped or thrown downward, the distance, d, that it falls in time, t, is described by the following equation:

$s = 4t^2 + v.t$

In this equation, v is the initial velocity, in m^2. Distance is measured in meters and time is measured in seconds. Use the equation to find how long it takes a tennis ball to reach the ground if it is thrown downward from a hot-air balloon that is 200 m high. The tennis ball is thrown at an initial velocity of $2m^2$.

Answer

$s = 4t^2 + v.t$

$200 = 4t^2 + 2.t$

$4t^2 + 2.t - 200 = 0$

$2t^2 + t - 100 = 0$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$t = \frac{-1 \pm \sqrt{(1)^2 - 4(2)(-100)}}{2(2)}$

$t = \frac{-1 \pm \sqrt{1+800}}{4}$

$t = \frac{-1 \pm \sqrt{801}}{4}$

$t = 6.83\ seconds \qquad or \qquad t = -7.33\ seconds$

Undefined

Therefore t = 6.83 seconds