GRADE 12
Calculus –Optimisation2.
WEBSITE NOTES

TOPIC:

- Practical problems concerning optimisation, rate of change and motion.

REMEMBER THAT A QUADRATIC FUNCTION WILL EITHER HAVE A MINIMUM OR MAXIMUM.

YOU NEED TO WORK OUT THE FUNCTION FROM INFORMATION GIVEN AND THEN OBTAIN THE DERIVATIVE.

TO GET THE MAXIMUM OR MINIMUM (OR BOTH IF CUBIC GRAPH), EQUATE THE DERIVATIVE TO 0.

Practical problems concerning optimisation, rate of change and motion continued

Example 1

The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this product a maximum.

Answer

Let the first number be x.
Let the second number be y.
Let the product be P.
We get the following two equations:

- ***The sum of two positive numbers is 20.***
  - $x + y = 20$
- ***One of the numbers is multiplied by the square of the other.***
  - ***xy^2 = Product***
  - $xy^2 = P$
- **Two Equations are**
  - $x + y = 20$................................ (1)
  - $P = xy^2$ .... (2)
- From (1)
  - $x = 20 - y$
- Substitute into (2)
  - $P = (20 - y).\,y^2$
  - $P = 20y^2 - y^3$
- Find the derivative and equate to 0
  - $0 = 40y - 3y^2$
  - $0 = y\,(40 - 3y)$
  - $0 = y$ or $0 = 40 - 3y = 0$
  - Therefore $y = 0$ or $-3y = -40$
  - Therefore $y = 0$ or $y = \frac{40}{3}$
  - NOTE y=0 can't be used because then we will have a minimum value for the product. ($x.0^2 = 0$)
- Substitute $y = \frac{40}{3}$ into (1)
  - $x + (\frac{40}{3}) = 20$

- $x = 20 - \frac{40}{3}$
- $x = \frac{60}{3} - \frac{40}{3}$
- $x = \frac{20}{3}$

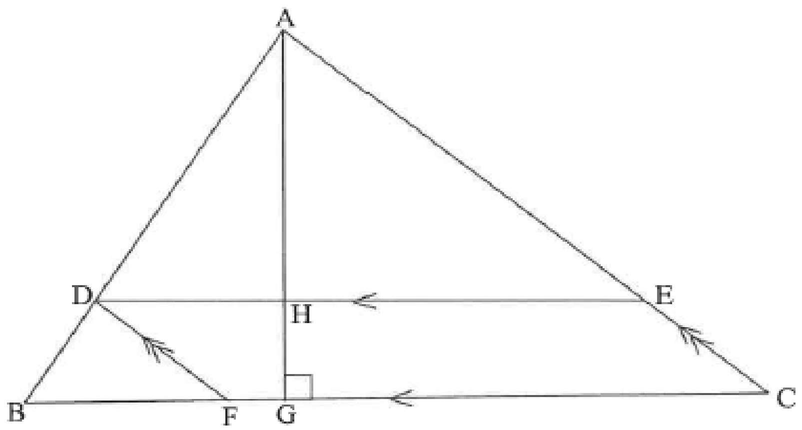
- Therefore, the values of the numbers are $x=\frac{20}{3}$ and $y=\frac{40}{3}$

Example 2 (Try yourself) PAST PAPER QUESTION

QUESTION 10

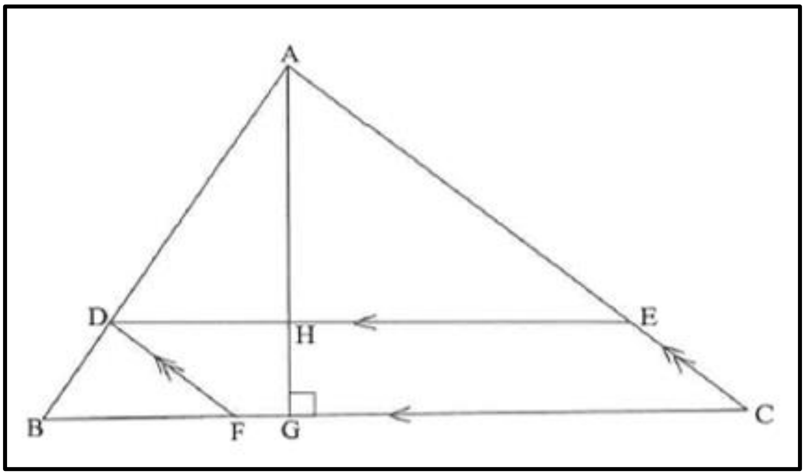
In ΔABC:

- D is a point on AB, E is a point on AC and F is a point on BC such that DECF is a parallelogram.
- BF : FC = 2 : 3.
- The perpendicular height AG is drawn intersecting DE at H.
- $AG = t$ units.
- $BC = (5 - t)$ units.



10.1	Write down AH : HG.	(1)
10.2	Calculate t if the area of the parallelogram is a maximum. (**NOTE:** Area of a parallelogram = base × $\perp$ height)	(5)
		[6]

Answer



10.1	$\frac{AH}{HG}=\frac{3}{2}$	✓ answer (1)

Explanation:

BF:FC = 2:3 (Given)

In Triangle ABC

Therefore BF:FC = BD: DA = 2:3 (Prop Theorem DF//AC)

In Triangle ABG

BD: DA = GH:HA = 2:3 (Prop Theorem DE//BC)

Therefore GH: HA = 2:3

Therefore HA:GH = 3:2

$$\frac{AH}{HG}=\frac{3}{2}$$

10.2	Area of a parallelogram $=$ base $\times \perp$ height Area $=\frac{3}{5}(5-t).\frac{2}{5}t$ Area $=\frac{6}{25}(5-t)t$ $A(t)=-\frac{6}{25}t^2+\frac{6}{5}t$ $A'(t)=-\frac{12}{25}t+\frac{6}{5}$ $-\frac{12}{25}t+\frac{6}{5}=0$ $12t-30=0$ $t=\frac{30}{12}$ or $\frac{5}{2}$	✓ $\frac{2}{5}t$ ✓ $\frac{3}{5}(5-t)$ ✓ $A(t)=-\frac{6}{25}t^2+\frac{6}{5}t$ ✓ $-\frac{12}{25}t+\frac{6}{5}$ ✓answer (5)
		[6]

Explanation:

BF:FC = 2:3 (Given)

BASE OF PARALLELOGRAM - FC

BC = 5-t

FC = $\frac{3}{5}(5-t)$

HEIGHT OF PARALLELOGRAM - HG

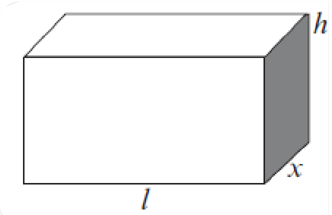
Therefore AH:HG = 3:2

AG = t

HG = $\frac{2}{5}(t)$

Example 3 (Try yourself)

A rectangular box is constructed in such a way that the length (l) of the base is three times as long as its width. The material used to construct the top and the bottom of the box costs R100 per square metre. The material used to construct the sides of the box costs R50 per square metre. The box must have a volume of 9 m^3. Let the width of the box be *x* metres.



3.1 Determine an expression for the height (*h*) of the box in terms of *x*.

3.2 Show that the cost to construct the box can be expressed as $C = \frac{1200}{x} + 600x^2$

3.3 Calculate the width of the box (that is the value of *x*) if the cost is to be a minimum.

ANSWER

3.1

Volume = *l . b . h*

9 = 3*x*.*x*.*h*

9 = $3x^2h$

$\frac{3}{x^2} = h$

3.2

C = (2(3*xh*) + 2*xh*) . 50 + (2 . $3x^2$) . 100

C = 300xh +100xh + $600x^2$

C = 400xh + $600x^2$

SUBSTITUTE $\frac{3}{x^2} = h$

C = 400x. $(\frac{3}{x^2})$ + $600x^2$

C = $\frac{1200}{x}$ + $600x^2$

3.3

C = $\frac{1200}{x}$ + $600x^2$

Remember to make sure that there is no variable in the denominator

C = $1200.x^{-1}$ + $600x^2$

FIND THE DERIVATIVE

$C' = -1200x^{-2} + 1200x$

MAKE THE DERIVATIVE = 0 TO GET THE MINIMUM AND SOLVE FOR X

$C' = -1200x^{-2} + 1200x$

$0 = -1200x^{-2} + 1200x$

$0 = \frac{-1200}{x^2} + 1200x$

$-1200x^3 = -1200$

$x^3 = \frac{-1200}{-1200} = 1$

$\therefore x = 1$

Example 4 (Try yourself)

A pump is connected to a public pool. The volume of the water is controlled by the pump and is given by the formula:

$$V(d) = 64 + 44d - 3d^2$$

where V = volume in kilolitres

d = days

4.1 Determine the rate of change of the volume of the reservoir with respect to time after 8 days.

4.2 Is the volume of the water increasing or decreasing at the end of 8 days? Explain your answer.

4.3 After how many days will the reservoir be empty?

4.4 When will the amount of water be at a maximum?

4.5 Calculate the maximum volume.

4.6 Draw a graph of V (d).

Answer

4.1 Rate of change = V ‘(d) ***(Derivative of V(d))***

V ‘(d) = 44 - 6d

After 8 days, rate of change will be:

V ‘(8) = 44 - 6(8)

= -4 kl per day

4.2 The rate of change is negative, so the function is decreasing.

4.3 Pool empty: V (d) = 0 ***(When the Volume is 0)***

$64 + 44d - 3d^2 = 0$

(16 - d) (4 + 3d) = 0

d = 16 or d = -4

It will be empty after 16 days

4.4 Maximum at turning point. (Work out the Derivative of V(d) and equate to 0)

Turning point where V ‘ (d) = 0

44 - 6d = 0

-6d = -44

$d = \frac{-44}{-6}$

$d = \frac{22}{3} = 7\frac{1}{3}$

$\therefore 7\frac{1}{3}\ days$

4.5 $V(d) = 64 + 44d - 6d^2$

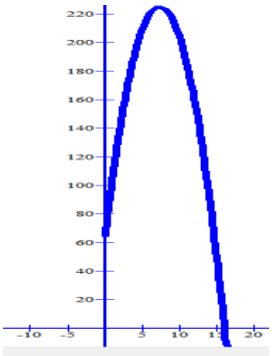
$V\left(7\frac{1}{3}\right) = 64 + 44(7\frac{1}{3}) - 6(7\frac{1}{3})^2$

Substitute d = $7\frac{1}{3}$ into the V(d) formula

$V\left(\frac{22}{3}\right) = 64 + 44(\frac{22}{3}) - 6(\frac{22}{3})^2$

$V\left(\frac{22}{3}\right) = 225{,}3\ kl$

4.6



Example 5 (Try yourself) ***(HINTS IN BOLD ITALICS)***

A soccer ball is kicked vertically into the air and its motion is represented by the equation:

$$D(t) = 1 + 18t - 3t^2$$

where D = distance above the ground (in metres)
t = time elapsed (in seconds)

5.1 Determine the initial height of the ball at the moment it is being kicked.
5.2 Find the initial velocity of the ball. ***(Velocity is the Derivative of the distance)***
5.3 Determine the velocity of the ball after 1,5 s.
5.4 Calculate the maximum height of the ball. ***(Derivative of the distance equals 0)***
5.5 Determine the acceleration of the ball after 1 second and explain the meaning of the answer.
(Acceleration is the second Derivative of the distance)
5.6 Calculate the average velocity of the ball during the third second.
0 to 1 second is the first second.
1 to 2 seconds is the second second.
2 to 3 seconds is the third second.
So, you need to work out the Average Distance between 2 to 3.
In other words, work out D (3) and D (2) and use the formula $\frac{D(3)-D(2)}{3-2}$
5.7 Determine the velocity of the ball after 3 seconds and interpret the answer.
5.8 How long will it take for the ball to hit the ground?
5.9 Determine the velocity of the ball when it hits the ground.

Answer

5.1
$D(t) = 1 + 18t - 3t^2$
$D(0) = 1 + 18(0) - 3(0)^2$
$= 1$ metre

5.2
Velocity = $D'(t) = 18 - 6t$
Initial velocity = $D'(0)$
$D'(0) = 18 - 6(0) = 18\ m.s^{-1}$

5.3
Velocity = $D'(t) = 18 - 6t$
Initial velocity = $D'(1,5)$
$D'(1,5) = 18 - 6(1,5) = 9\ m.s^{-1}$

5.4
Maximum height is at the turning point. It is where the velocity is 0. Where it starts coming back to ground.
$D'(t) = 18 - 6t$
$0 = 18 - 6t$
$-18 = -6t$
$t = \frac{-18}{-6} = 3$
Substitute t = 3 into the distance formula, $D(t) = 1 + 18t - 3t^2$
$D(3) = 1 + 18(3) - 3(3)^2$
$D(3) = 28$ metres

5.5
Acceleration = $D''(t)$
$D''(t) = -6\ m.s^{-2}$
The velocity is decreasing by 6 metres per second per second.

5.6
$\frac{D(3)-D(2)}{3-2} = \frac{(1+18(3)-3(3)^2)-(1+18(2)-3(2)^2)}{1} = (1 + 18(3) - 3(3)^2) - (1 + 18(2) - 3(2)^2) = 3m.s^{-1}$

5.7
$D'(3) = 18 - 6(3)$
$D'(3) = 0m.s^{-1}$
This is the stationary point, where the derivative is zero. The ball has stopped going up and is about to begin its descent.

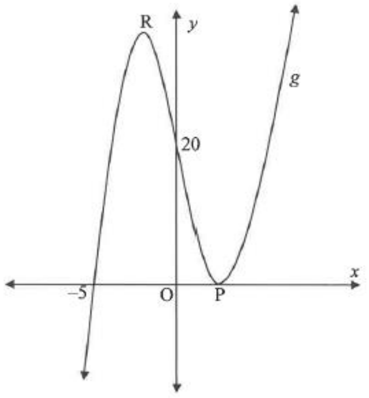
5.8
Hits ground: $D(t) = 0$
$D(t) = 1 + 18t - 3t^2$
$t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$t = \frac{-18\pm\sqrt{(18)^2-4(-3)(1)}}{2(-3)}$
$t = \frac{-18\pm\sqrt{336}}{-6}$
$t = -0{,}05$ or $t = 6{,}05$
The ball hits the ground at 6,05 s (time cannot be negative).

5.9
$D'(t) = 18 - 6t$
When t = 6,05 it will be the velocity when the ball hits the ground.
$D'(6{,}05) = 18 – 6(6{,}05)$
$D'(6{,}05) = 18{,}3m.s^{-1}$

Example 6 (Try yourself) PAST PAPER QUESTION ON CUBIC FUNCTIONS

QUESTION 9

9.1 The graph of $g(x) = x^3 + bx^2 + cx + d$ is sketched below.
The graph of g intersects the x-axis at $(-5\ ;\ 0)$ and at P, and the y-axis at $(0\ ;\ 20)$.
P and R are turning points of g.



9.1.1 Show that $b = 1$, $c = -16$ and $d = 20$. (4)

9.1.2 Calculate the coordinates of P and R. (5)

9.1.3 Is the graph concave up or concave down at $(0\ ;\ 20)$? Show ALL your calculations. (3)

Answer

QUESTION/VRAAG 9

9.1.1	$g(x) = (x+5)(x-x_1)^2$ $20 = 5(x_1)^2$ $x_1^2 = 4$ $x_1 = 2$ $g(x) = (x+5)(x-2)^2$ $g(x) = (x+5)(x^2-4x+4)$ $g(x) = x^3 + x^2 - 16x + 20$	✓ $(x+5)$ ✓repeated root ✓$x_1 = 2$ ✓ $g(x) = (x+5)(x^2-4x+4)$ (4)
9.1.2	$g(x) = x^3 + x^2 - 16x + 20$ $g'(x) = 3x^2 + 2x - 16$ $3x^2 + 2x - 16 = 0$ $(3x+8)(x-2) = 0$ $x = \frac{-8}{3}$ or $x = 2$ $\text{R}\left(\frac{-8}{3};\frac{1372}{27}\right)$ or $\text{R}(-2{,}67;50{,}81)$ $\text{P}(2;0)$	✓derivative ✓equating to zero ✓factors ✓co-ordinates of R ✓ co-ordinates of P (5)
9.1.3	$g''(x) = 6x + 2$ $g''(0) = 2$ $\therefore$ concave up **OR/OF** $g''(x) = 6x + 2$ $6x + 2 = 0$ $x = -\frac{1}{3}$ is the point of inflection $\therefore$ concave up	✓ $g''(x) = 6x + 2$ ✓ $g''(0) = 2$ ✓conclusion (3) **OR/OF** ✓ $g''(x) = 6x + 2$ ✓ $x = -\frac{1}{3}$ ✓conclusion (3)

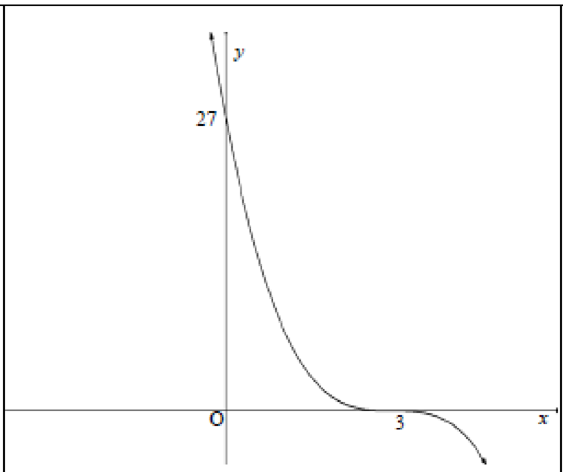
Example 7 (Try yourself) PAST PAPER QUESTION ON CUBIC FUNCTIONS

9.2 If g is a cubic function with:

- $g(3) = g^{/}(3) = 0$
- $g(0) = 27$
- $g^{//}(x) > 0$ when $x < 3$ and $g^{//}(x) < 0$ when $x > 3$,

draw a sketch graph of g indicating ALL relevant points. (3)

Answer

9.2		✓ y – intercept of a cubic graph ✓ point of inflection and stationary point, $x = 3$ ✓ concave up for $x < 3$ and concave down for $x > 3$ (3)
		[15]