GRADE 11
Trigonometry Identities and GENERAL SOLUTIONS_2
WEBSITE NOTES 4

TOPIC:

- Derivation and use of the identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$
- Derivation and use of reduction formulae for $\sin(90^0 \pm \theta)$, $\cos(90^0 \pm \theta)$, $\sin(180^0 \pm \theta)$, $\cos(180^0 \pm \theta)$, $\tan(180^0 \pm \theta)$, $\sin(360^0 \pm \theta)$, $\cos(360^0 \pm \theta)$, $\tan(360^0 \pm \theta)$ $\sin(-\theta)$, $\cos(-\theta)$, $\tan(-\theta)$
- ***Determine the general solution and / or specific solutions (given intervals) of trigonometric equations. (NEW TOPIC)***

Example 1 (Try Yourself – using identities)

Prove the following identities:

1. $\sin x \cdot \tan x + \cos x = \frac{1}{\cos x}$ (4)

2. $(\sin x + \tan x)\left(\frac{\sin x}{1+\cos x}\right) = \sin x.\tan x$ (7)

3. $\frac{1}{\cos x} = \frac{\cos x}{1+\sin x} + \tan x$ (6)

4. $\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$ (5)

[22]

Answers

1. LHS: $\sin x.\tan x + \cos x$

$= \sin x . \frac{\sin x}{\cos x} + \cos x$ ✓ $+ \cos x$

$= \frac{\sin^2 x}{\cos x} + \frac{\cos x}{1}$

$= \frac{\sin^2 x + \cos^2 x}{\cos x}$ ✓✓ $= \frac{1}{\cos x}$ ✓ = RHS (4)

$\therefore \sin x.\tan x + \cos x = \frac{1}{\cos x}$ (4)

2. LHS: $(\sin x + \tan x)\left(\frac{\sin x}{1+\cos x}\right)$

$= \left(\sin x + \frac{\sin x}{\cos x}\right)$ ✓ $\left(\frac{\sin x}{1+\cos x}\right)$

$= \left(\frac{\sin x\cos x + \sin x}{\cos x}\right)$ ✓✓ $\left(\frac{\sin x}{1+\cos x}\right)$

$= \left(\frac{\sin x(\cos x + 1)}{\cos x}\right)$ ✓ $\left(\frac{\sin x}{1+\cos x}\right)$

$= \frac{\sin^2 x}{\cos x}$ ✓ (7)

RHS: $\sin x.\tan x$

$= \sin x.\frac{\sin x}{\cos x}$ ✓

$= \frac{\sin^2 x}{\cos x}$ ✓

$\therefore$ LHS = RHS (7)

3. RHS: $\frac{\cos x}{1+\sin x} + \tan x$

$= \frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x}$ ✓

$= \frac{\cos^2 x + \sin x(1+\sin x)}{\cos x(1+\sin x)}$ ✓ ✓

$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1+\sin x)}$ ✓ trig identity: $\cos^2 x + \sin^2 x = 1$

$= \frac{1+\sin x}{\cos x(1+\sin x)}$ ✓

$= \frac{1}{\cos x}$ ✓ = LHS

$\therefore \frac{1}{\cos x} = \frac{\cos x}{1+\sin x} + \tan x$ (6)

4. $\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$

LHS: $\frac{1}{\tan x} + \tan x$ RHS: $\frac{\tan x}{\sin^2 x}$

$= \frac{1}{\frac{\sin x}{\cos x}} + \frac{\sin x}{\cos x}$ ✓ $= \frac{\sin x}{\cos x}$ ✓ $\cdot \frac{1}{\sin^2 x}$

$= \frac{\cos x}{\sin x}$ ✓ $+ \frac{\sin x}{\cos x}$ $= \frac{1}{\sin x.\cos x}$

$= \frac{\cos^2 x + \sin^2 x}{\sin x.\cos x}$ ✓ ✓

$= \frac{1}{\sin x.\cos x}$

$\therefore$ LHS = RHS (5)

[22]

HINTS

- Choose either the **left-hand side** or the **right-hand side** and simplify it to look like the other side.
- If both sides look difficult, you can try to simplify on both sides until you reach a point where both sides are the same.
- It is usually helpful to write $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.
- Sometimes you need to simplify $\frac{\sin\theta}{\cos\theta}$ to $\tan\theta$.
- If you have $\sin^2 x$ or $\cos^2 x$ with +1 or −1, use the squares identities ($\sin^2\theta + \cos^2\theta = 1$).
- Find a common denominator when fractions are added or subtracted.
- Factorise if necessary

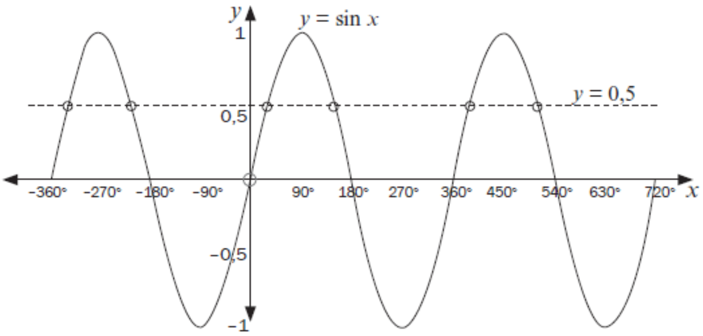
General Solution of Trig Equations

When you solve Trig Equations, you will put it in General Solution form for the domain indicated because there can be more than 1 possible answer.

To solve a trig equation where the angle is unknown, you need to find all the possible values of the angle.

For example, if $\sin\theta = \frac{1}{2}$, we know that θ could be $30°$. However, there are other values for θ in the other quadrants. Have a look at the graph for sin $\theta = \frac{1}{2}$, $\theta \in [-360°; 720°]$.

There are six values for θ between $-360°$ and $720°$.



If $30°$ is our reference angle in quadrant I.

In quadrant II: $\sin(180° - 30°) = \sin 30° = \frac{1}{2}$ So θ is $150°$

In quadrant III and IV, the sine ratio is negative, so there is no solution for θ.

The angle could be greater than $360°$.

In quadrant I: $\sin(360° + 30°) = \sin 30° = \frac{1}{2}$ So θ is $390°$

In quadrant II: $\sin(540° - 30°) = \sin((540° - 360°) - 30°)$
$= \sin(180° - 30°) = \sin 30 = \frac{1}{2}$ So θ is $510°$

You can also work out that $\theta = -210°$ or $\theta = -330°$

You do not need to draw a graph to solve these equations.

Example 1

1. Solve for x: $\sin x = 0{,}7$ [On your calculator, press: $\sin^{-1}$ 0,7 =]
The calculator answer is 44,42……°
We call this the **reference angle**, as it is not the only solution to the equation.
$\sin x$ is positive, so angle x must be in quadrant I or quadrant II in the first revolution.
In quadrant I: $x = 44{,}42……$ °
AND
In quadrant II: $x = 180° - 44{,}42….° = 135{,}57……..$ °
The period of the sin graph is 360°, so the other points of intersection occur 360° to the right or left of these solutions.
We add k revolutions to the two angles in the first revolution.
k is an integer (…–1; 0; 1; …). We call this **the general solution** of the equation.
So we can say the solution to **sin x = 0,7** is
$x = 44{,}42° + k360°$ or $x = 135{,}57° + k360°$; $k \in \mathbb{Z}$.
(Correct to two decimal place)

Example 2

2. Solve for x: $\sin x = -0{,}7$
This time, place the reference angle in quadrants III and IV ($\sin x$ is negative)
$x = 180° + 44{,}42…..° + k360°$ or $x = 360° - 44{,}42…..° + k360°$ $k \in \mathbb{Z}$
$x = 224{,}42° + k360°$ or $x = 315{,}57° + k360$; $k \in \mathbb{Z}$
(Correct to two decimal place)

Example 3

3. Solve for x: $\cos x = -0{,}7$ Reference angle = 134,427….°
$\cos x$ is negative in quadrants II and III.
$x = 360° - 134{,}43° = 225{,}57°$
$x = 134{,}43° + k360°$ or $x = 225{,}57° + k360°$; $k \in \mathbb{Z}$
(Correct to two decimal place)

Example 4

4. Solve for x: $\cos x = 0{,}7$ Reference angle = 45,57……°
This time, place the reference angle in quadrants I and IV where $\cos x$ is positive:
$x = 45{,}57….° + k360°$ or $x = 360° - 45{,}57……° + k360°$
$x = 45{,}57° + k360°$ or $x = 314{,}43° + k360°$; $k \in \mathbb{Z}$.
(Correct to two decimal place)

Example 5

5. Solve for x: $\tan x = 0,7$

tan x is positive in quadrants I and III.

Reference angle = 34,99° (correct to 2 dec places)

$x = 34,99....°$ or $180° + 34,99......° = 214,99.......°$

Now the period of the tan graph is 180°, so the other points of intersection occur 180° to the right or left of the solutions.

$x = 34,99° + k180°; k \in \mathbb{Z}$

(Correct to two decimal place)

NB!!!

For Tan equations general solution, we use $+k.180^0$ because the period of a Tan function is 180^0

Example 6

6. Solve for x: $\tan x = -0,7$

tan x is negative in quadrants II and IV.

The reference angle is $-34,99......°$

$180° - 34,99.......° = 145,01.....°$

$x = 145,01° + k180°; k \in \mathbb{Z}$.

Example 7 (try yourself)

Determine the general solution for x in the following equations:

a) $5 \sin x = \cos 320°$ (correct to 2 decimal places)

b) $3 \tan x + \sqrt{3} = 0$ (without using a calculator)

c) $\frac{\tan x - 1}{2} = -3$ (correct to one decimal place) (10)

Answers

a) $5 \sin x = \cos 320°$ ✓

$5 \sin x = 0,766044$

$\sin x = 0,15320...$ ✓

Ref angle = 8,81°

$x = 8,81° + k360°$ OR $x = 180° - 8,81° + k360°$ ✓

$x = 171,19° + k360°$ ✓ $k \in \mathbb{Z}$ (4)

Calculator keys:
cos 320 =
÷ 5 =
SHIFT sin ANS =

b) $3 \tan x + \sqrt{3} = 0$

$3 \tan x = -\sqrt{3}$

$\tan x = \frac{-\sqrt{3}}{3}$ ✓ [special angle: tan 30° $\tan 30° = \frac{\sqrt{3}}{3}$]

Ref angle = 30°

$x = 180° - 30° + k180°$ ✓

$x = 150° + k180°$ ✓ $k \in \mathbb{Z}$ (3)

$\frac{\tan x-1}{2} = -3$ multiply both sides by 2

$\tan x - 1 = -6$

$\tan x = -5$ ✓ reference angle is 78,69…°

$\therefore x = 180° - 78,69…° + k180°$ ✓

$x = 101,31° + k180°; k \in \mathbb{Z}$ ✓ (3) (10)