GRADE 11
Trigonometry Identities_1
WEBSITE NOTES 3

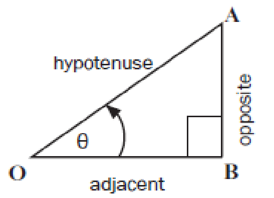
TOPIC:

- Derivation and use of the identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$
- Derivation and use of reduction formulae for sin$(90^0 \pm \theta)$, cos $(90^0 \pm \theta)$, sin $(180^0 \pm \theta)$, cos $(180^0 \pm \theta)$, tan $(180^0 \pm \theta)$, sin $(360^0 \pm \theta)$, cos $(360^0 \pm \theta)$, tan $(360^0 \pm \theta)$ sin $(-\theta)$, cos $(-\theta)$, tan $(-\theta)$

Revision of Trig ratios

The trigonometric ratios
Using θ as the **reference angle** in ΔABO

- The side opposite the 90° is the hypotenuse side, therefore side AO is the hypotenuse side.
- The side opposite θ is the opposite side, therefore AB is the opposite side.
- The side adjacent to θ is called the adjacent side, therefore OB is the adjacent side.



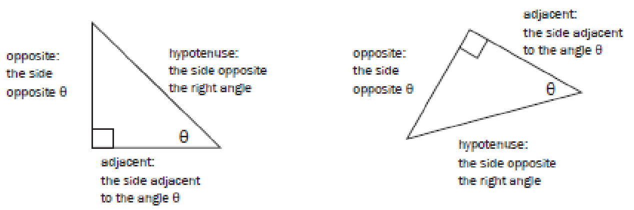
We work with the ratios of the sides of the triangle:

- The ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is called **sine** θ (abbreviated to **sin θ**)
- The ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called **cosine** θ (abbreviated to **cos θ**)
- The ratio $\frac{\text{opposite}}{\text{adjacent}}$ is called **tangent** θ (abbreviated to **tan θ**)

Therefore $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = AB/AO$

$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = OB/AO$

$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = AB/OB$

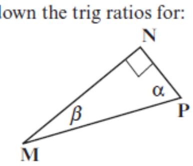


LEARN THESE!!!!!!!

$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$

Example 1

1. ΔMNP is a right-angled triangle. Write down the trig ratios for:
   a) sin α b) sin β (4)
   c) tan β d) cos α
2. If MP = 13 and NP = 5, calculate cos β. (3)



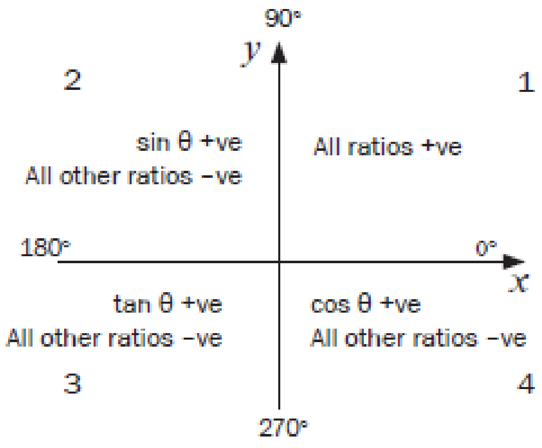
[7]

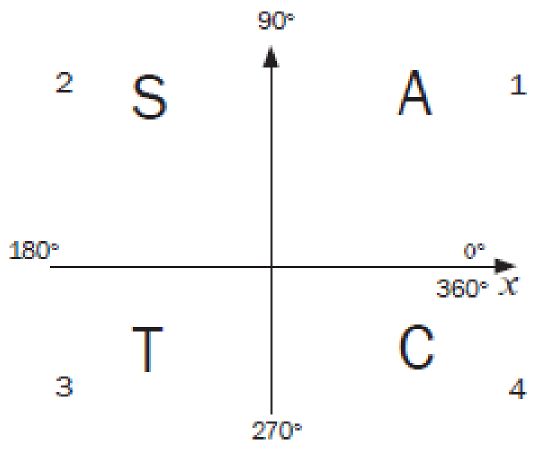
Answer

1. a) $\sin\alpha = \frac{MN}{MP}$ ✓ (1) b) $\sin\beta = \frac{NP}{MP}$ ✓ (1)
   c) $\tan\beta = \frac{NP}{MN}$ ✓ (1) d) $\cos\alpha = \frac{NP}{MP}$ ✓ (1) (4)
2. MP = 13 and NP = 5, so we can find MP,
   $MP^2 = MN^2 + NP^2$ ....Pythagoras ✓
   $13^2 = MN^2 + 5^2$
   $169 = MN^2 + 25$
   $MN^2 = 169 - 25$
   $MN^2 = 144$ ✓
   $\therefore MN = 12$
   $\cos\beta = \frac{MN}{MP} = \frac{12}{13}$ ✓ (3)

[7]

Trig Ratios in each quadrant of Cartesian Plane





Example 2

1. If sin θ is negative and cos θ is positive, then which statement is true?
   A. $0° < \theta < 90°$ B. $90° < \theta < 180°$
   C. $180° < \theta < 270°$ D. $270° < \theta < 360°$ (1)
2. If $\tan \theta < 0$ and $\cos \theta < 0$, then which statement is true?
   A. $0° < \theta < 90°$ B. $90° < \theta < 180°$
   C. $180° < \theta < 270°$ D. $270° < \theta < 360°$ (1)
3. Will the following trig ratios be positive or negative?
   a) sin 315°
   b) cos (–215°)
   c) tan 215°
   d) cos 390° (4)

[6]

Answer

1. Sin θ is negative in 3rd and 4th quadrants; cos θ is positive in 1st and 4th quadrants.
   So θ is in the 4th quadrant. **D.** $270° < \theta < 360°$ ✓ (1)
2. $\tan \theta < 0$ in 2nd and 4th quadrants; $\cos \theta < 0$ in 2nd and 3rd quadrants.
   So θ is in the 2nd quadrant. **B.** $90° < \theta < 180°$ ✓ (1)
3. a) sin 315° is in 4th quadrant so it is negative. ✓ (1)
   b) cos (–215°) is in 2nd quadrant so it is negative. ✓ (1)
   c) tan 215° is in 3rd quadrant, so it is positive. ✓ (1)
   d) cos 390° is the same as cos 30° in the 1st quadrant, so it is positive. ✓ (1)

[6]

Identities

QUOTIENT IDENTITY

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

SQUARE IDENTITY

$\sin^2\theta + \cos^2\theta = 1$

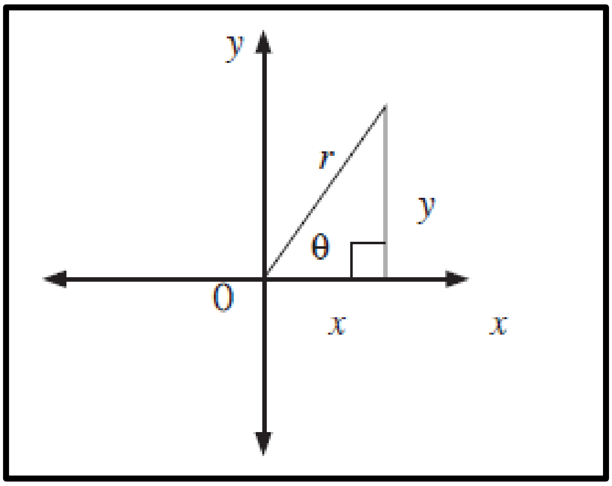
From the above we can derive the following:

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

PROOF OF THE ABOVE IDENTITIES (NEED TO KNOW)

Write the proofs out a couple of times until you can do them without looking at the notes.



Proof of the identities are examinable with the RHS and break it down into its x, y and r values.

Proof: $\frac{\sin\theta}{\cos\theta}$

$= \frac{y}{r} \div \frac{x}{r}$

$= \frac{y}{r} \times \frac{r}{x}$

$= \frac{y}{x} = \tan\theta$

Proof: $\sin^2\theta + \cos^2\theta$

$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$

$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$ Use LCD r^2

$= \frac{x^2 + y^2}{r^2}$ $x^2 + y^2 = r^2$ (Pythagoras)

$= \frac{r^2}{r^2} = 1$

We can use the identities and the reduction formulae to help us simplify trig expression

Special Angles

These will always be valid for 30^0; 45^0; 60^0. You must learn and remember the special angles ratios. It is used with questions that say **WITHOUT THE USE OF A CALCULATOR**.

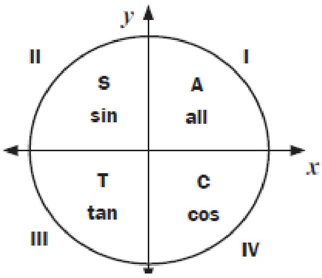
θ	30°	45°	60°
sin θ	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan θ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Reduction Formula

We use Reduction Formula to simplify the trig function or expression to obtain an acute angle.

a) Reduction formulae

Quadrant II: 180° − θ	Quadrant III: 180° + θ	Quadrant IV: 360° − θ
sin(180° − θ) = sin θ	sin(180° + θ)= −sin θ	sin(360° − θ) = −sin θ
cos(180° − θ) = −cos θ	cos(180° + θ) = −cos θ	cos(360° − θ) = cos θ
tan(180° − θ) = −tan θ	tan(180° + θ) = tan θ	tan(360° − θ) = −tan θ



b) **Angles greater than 360°**
We can add or subtract 360° (or multiples of 360°) and will always end up with an angle in the first revolution. For example, 390° can be written as (30° + 360°), so 390° has the same terminal arm as 30°.

c) **Negative angles:**

- (−θ) lies in quadrant IV and is the same as 360° − θ.

sin(−θ) = −sin θ	cos(−θ) = cos θ	tan(−θ) = −tan θ

Example 3

$\sin(360° + \theta) = \sin\theta$ $\quad$ $\cos(360° + \theta) = \cos\theta$ $\quad$ $\tan(360° + \theta) = \tan\theta$

Example 4

In this example the reduction formula is used to obtain acute angles.
We always try first to obtain an acute angle that is one of the special angles first if possible.

Without using a calculator, determine the value of:

1. cos 150° **2.** sin (–45°) **3.** tan 480°

[7]

Solutions

1. cos 150°	rewrite as (180 – ?)
$= \cos(180° - 30°)$	quadrant II, cos θ negative
$= -\cos 30°$ ✓	special ratios
$= -\frac{\sqrt{3}}{2}$ ✓ (2)	
2. sin(–45°)	sin(–θ) = –sin θ; quadrant IV, sin θ negative
$= -\sin 45°$ ✓	special ratios
$= -\frac{1}{\sqrt{2}}$ ✓ (2)	
3. tan 480°	write as an angle in the first rotation of 360°
$= \tan(480° - 360°)$	
$= \tan 120°$ ✓	quadrant II, rewrite as (180 – ?)
$= \tan(180° - 60°)$	tan θ negative
$= -\tan 60°$ ✓	special ratios
$= -\sqrt{3}$ ✓ (3)	[7]

Co-Functions

The functions change from cos to sin or sin to cos if we use 90^0+ or 90^0- to reduce. The signs of whether the function is positive or negative may also change.

$\sin(90° - \theta) = \cos\theta$ (quadrant I)

$\sin(90° + \theta) = \cos\theta$ (sin θ positive in quadrant II)

$\cos(90° - \theta) = \sin\theta$ (quadrant I)

$\cos(90° + \theta) = -\sin\theta$ (cos θ negative in quadrant II)

SOME OTHER POSSIBILITIES THAT COULD COME UP

IT NEEDS TO BE 90^0+ or 90^0- in order to use the co-function reduction formula

$\sin(\theta - 90°) = \sin[-(90° - \theta)]$ (common factor of –1)

$= -\sin(90° - \theta)$ (sin θ negative in quadrant IV)

$= -\cos\theta$

$\cos(\theta - 90°) = \cos[-(90° - \theta)]$ (common factor of –1)

$= +\cos(90° - \theta)$ (cos θ positive in quadrant IV)

$= +\sin\theta$

Example 5

Write the trig ratios as the trig ratios of their co-functions:

1. sin 50° **2.** cos 70° **3.** sin 100° **4.** cos 140°

[4]

Solutions

1. $\sin 50° = \sin(90° - 40°) = \cos 40°$ ✓
2. $\cos 70° = \cos(90° - 20°) = \sin 20°$ ✓
3. $\sin 100° = \sin(90° + 10) = \cos 10°$ ✓
4. $\cos 140° = \cos(90° + 50°) = -\sin 50°$ ✓

[4]

NOT ALWAYS SPECIAL ANGLES

SUMMARY

Any angle (obtuse or reflex) can be reduced to an acute angle by using:

- Convert negative angles to positive angles
- Reduce angles greater than 360°
- Use reduction formulae
- Use co-functions

Example 6 (Try Yourself)

Simplify without using a calculator:

1. $\dfrac{\sin(180° + x).\cos 330°.\tan 150°}{\sin x}$ (4)

2. $\dfrac{\cos 750°.\tan 315°.\cos(-\theta)}{\cos(360° - \theta).\sin 300°.\sin(180° - \theta)}$ (8)

3. $\dfrac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 225°}$ (9)

4. $\dfrac{\cos 260°.\cos 170°}{\sin 10°.\sin 190°.\cos 350°}$ (7)

[28]

Answers

1. $\dfrac{\sin(180° + x).\cos 330°.\tan 150°}{\sin x}$ — reduction formulae in numerator

✓ ✓ ✓

$= \dfrac{(-\sin x)(+\cos 30°)(-\tan 30°)}{\sin x}$ — (use brackets to separate ratios)

$= \dfrac{+\sin x.\frac{\sqrt{3}}{2}✓.\frac{\sqrt{3}}{3}}{\sin x}$ — special angles

$= \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{3}$

$= \dfrac{3}{6} = \dfrac{1}{2}$ (4)

2. $\dfrac{\cos 750°.\tan 315°.\cos(-\theta)}{\cos(360° - \theta).\sin 300°.\sin(180° - \theta)}$ — use reduction formulae

$= \dfrac{\cos 30° ✓.(-\tan 45°) ✓.\cos\theta ✓}{\cos\theta ✓.(-\sin 60°) ✓.\sin\theta ✓}$ — use special angles

$= \dfrac{\frac{\sqrt{3}}{2}.(-1)\cos\theta}{\cos\theta.\left(-\frac{\sqrt{3}}{2}\right)\sin\theta}$ ✓

$= \dfrac{-1}{-\sin\theta} = \dfrac{1}{\sin\theta}$ ✓ (8)

3. $\dfrac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 225°}$

$= \dfrac{\tan 120°.(-\sin 60) ✓.\cos 14°.\sin 225°}{\sin 76° ✓.(-\cos 45°) ✓}$

$= \dfrac{\cos(180° + 80°).\cos(180° - 10°)}{\sin 10°.\sin(180° + 10°).\cos(360° - 10°)}$

$= \dfrac{(-\tan 60°)✓.(-\sin 60°).\sin 76°✓.(-\sin 45°)✓}{\sin 76°.(-\cos 45°)}$

$= \dfrac{(-\sqrt{3}).\left(\frac{-\sqrt{3}}{2}\right).\sin 76.\left(\frac{-\sqrt{2}}{2}\right)}{\sin 76°.\left(\frac{-\sqrt{2}}{2}\right)}$ ✓✓

$= \dfrac{3}{2}$ ✓ (9)

4. $\dfrac{\cos 260°.\cos 170°}{\sin 10°.\sin 190°.\cos 350°}$

$= \dfrac{-\cos 80°✓.(-\cos 10°)}{\sin 10°.(-\sin 10°)✓.\cos 10°✓}$

$= \dfrac{(-\sqrt{3}).\left(\frac{-\sqrt{3}}{2}\right).\sin 76.\left(\frac{-\sqrt{2}}{2}\right)}{\sin 76°.\left(\frac{-\sqrt{2}}{2}\right)}$ ✓✓

$= \dfrac{-\sin 10° ✓.(-\cos 10°)}{\sin 10°.(-\sin 10°).\cos 10°}$

$= \dfrac{-1}{\sin 10°}$ ✓ (7)

[28]