GRADE 12

Analytical Geometry

WEBSITE NOTES 3

TOPIC: The equation of a circle (any centre)

$$(x-a)^2+(y-b)^2=r^2$$

The equation of a tangent to a circle

In Grade 11 you learnt:

1. Distance Formula
   $AB=\sqrt{(x_a-x_b)^2+(y_a-y_b)^2}$
2. Gradient between two points
   $mAB=\frac{y_a-y_b}{x_a-x_b}$
3. The MIDPOINT between two points
   Midpoint AB $=(\frac{x_a+x_b}{2};\frac{y_a+y_b}{2})$
4. m = tan A (where m is the gradient of a line and A is the angle of inclination)

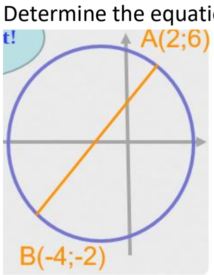
Revise to work out the equation of a circle

Diameter is 2 x radius

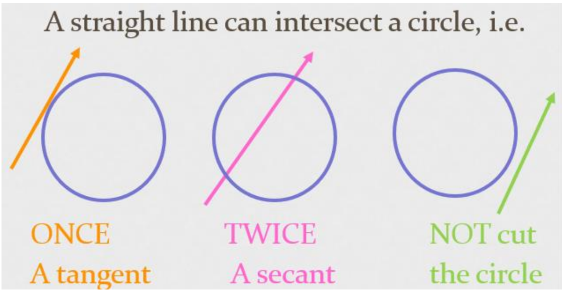
Examples to try

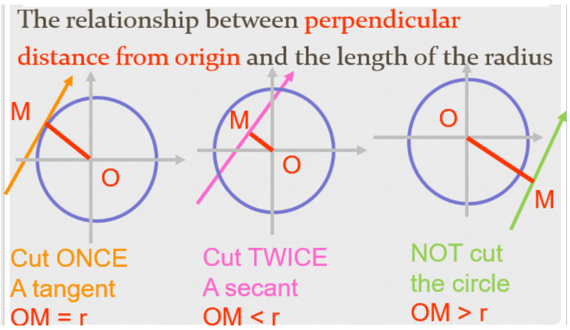
1. ***Example where you are given the centre and the radius***
   Find the general equation a circle centre (-1;3) with a radius of 5 units.

2. ***Example where you are given the Diameter***
   Determine the equation the circle, given the diameter AB



Tangents





Where r is the radius

Example 1

Show that $2x + 5 + y = 0$ is a tangent to the circle $x^2 + y^2 = 5$.

You will need to use simultaneous equations in this type of example. WHY???

You need to show that there is only 1 point of intersection. So you need to work out for x or for y. Look at the equations given and decide on the easier one to work out.

In this example $2x + 5 + y = 0$ can be easily changed to $y = -2x - 5$ and then substituted into $x^2 + y^2 = 5$.

$y = -2x - 5$ ……… (1) $x^2 + y^2 = 5$ ……….. (2)

Substitute (1) into (2)

$x^2 + (-2x - 5)^2 = 5$

We are going to solve then only for x. If the x-values are the same, then the y-values will also be the same if we substitute back.

$x^2 + (-2x - 5)^2 = 5$

$x^2 + (4x^2 + 20x + 25) = 5$

$5x^2 + 20x + 20 = 0$

$x^2 + 4x + 4 = 0$

$(x + 2)(x + 2) = 0$

$x = -2$ or $x = -2$

Since there is only 1 point of intersection (i.e. x= -2), the line is a TANGENT

If 2 DIFFERENT points of intersection are found, the line is a SECANT.

If there is no solution, the line does not intersect the circle.

Example 2

Find the equation of the tangent to the circle $x^2 + y^2 = 5$ at the point $(-2; 1)$.

REMEMBER:

Radius is ⊥ to Tangent

$m_{radius} \times m_{tangent} = -1$

m being the gradient

Answer

Tangent: $y = mx + c$

Tangent is a straight line, so we use the equation of a straight line

$m_{radius} = \frac{1-0}{-2-0} = \frac{-1}{2}$

Work out the gradient of the radius. The centre of the circle is (0;0).

Radius is ⊥ to Tangent

$m_{radius} \times m_{tangent} = -1$

$\therefore m_{tangent} = 2$

$y = 2x + c$

Solve for c now.

Substitute point of contact of tangent:

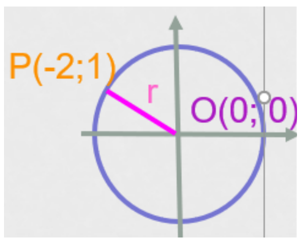
Subst. P(-2; 1)

$1 = 2(-2) + c$

$c = 5$

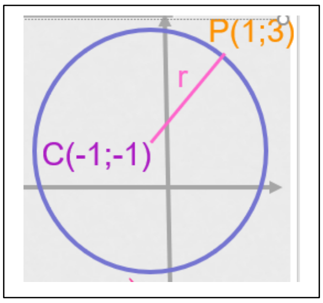
$\therefore$ *Equation of tangent is:*

$y = 2x + 5$



Example 3 (Try yourself)

Find the equation of the tangent at the point P(1;3), given the centre C(-1;-1).



ANSWER

Tangent: y = mx + c

$m_{radius} = \frac{3-(-1)}{1-(-1)} = 2$

$m_{radius} \times m_{tangent} = -1$

$\therefore m_{tangent} = \frac{-1}{2}$ (radius ⊥ tangent)

$\therefore y = \frac{-1}{2}x + c$

Subst. P(1; 3)

$3 = \frac{-1}{2}(1) + c$

$c = \frac{7}{2}$

$\therefore Equation\ of\ tangent\text{: } y = \frac{-1}{2}x + \frac{7}{2}$

Example 4 (Try yourself)

Find the equation of the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 16$ at (6;0).

HINT: the centre of the circle you can get from the circle formula.

Answer

Centre: (2;3)

$m_{radius} = \frac{3-0}{2-6} = \frac{-3}{4}$

$m_{radius} \times m_{tangent} = -1$ (radius ⊥ tangent)

$\therefore m_{tangent} = \frac{4}{3}$

$\therefore y = \frac{4}{3}x + c$

Subst. P(6; 0)

$0 = \frac{4}{3}(6) + c$

$c = -8$

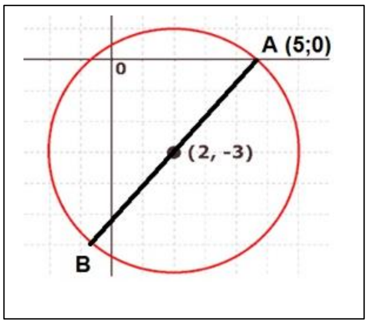
$\therefore Equation\ of\ tangent\ is\text{:}$

$y = \frac{4}{3}x + 8$

Work out the co-ordinates of B using midpoint theorem

Example 5 (Try yourself)

Given the circle with diameter AB and centre (1;-1).



1. Find the equation of the circle.

Answer

1.

$\frac{x+5}{2} = 2 \quad \frac{y+0}{2} = -3$

$x + 5 = 4 \quad y + 0 = -6$

$x = -1 \quad y = -6$

$\therefore$ B(− 1; −6)

$(x - a)^2 + (y - b)^2 = r^2$

$(x - 2)^2 + (y - (-3))^2 = r^2$

Subst. A(5; 0)

$(5 - 2)^2 + (0 + 3)^2 = r^2$

$r^2 = 18$

$\therefore (x - 2)^2 + (y + 3)^2 = 18$

2. The equation of the tangent at A

$m_{radius} = \frac{0-(-3)}{5-2} = 1$
$\therefore m_{tangent} = -1$
(radius $\perp$ tangent)
$y = -x + c$
$0 = -(5) + c$
$c = 5$

$\therefore y = -x + 5$

3. Determine the equation of a line parallel to the tangent at A and passing through the point C (2;8).

Tangent at A: $y = -x + 5$
Parallel line: $y = -x + c$
$8 = -(2) + c$
$c = 10$
$\therefore y = -x + 10$

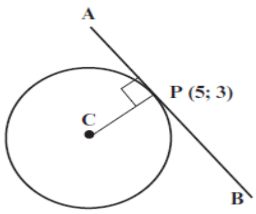
Parallel Lines have equal gradients

4. Determine whether the tangent at B will intersect with the tangent at A.
HINT: ***Work out the equation of the Tangent at B and equate Tangent at A to Tangent at B.***

Tangent at A: $y = -x + 5$
Tangent at B$(-1;-6)$:
$m_{radius} = \frac{-6-(-3)}{-1-2} = 1$
$\therefore m_{tangent} = -1$
(radius $\perp$ tangent)
Tangent at B$(-1;-6)$:
$y = -x + c$
$-6 = -(-1) + c$
$c = -7$
$\therefore y = -x - 7$
$\therefore -x + 5 = -x - 7$
$0 = -12$ ... *No solution*
Tangents do not intersect

Example 6 (Try yourself)
Find the equation of the tangent APB which touches a circle centre C with equation $(x-3)^2 + (y+1)^2 = 20$ at P(5; 3).
Answer

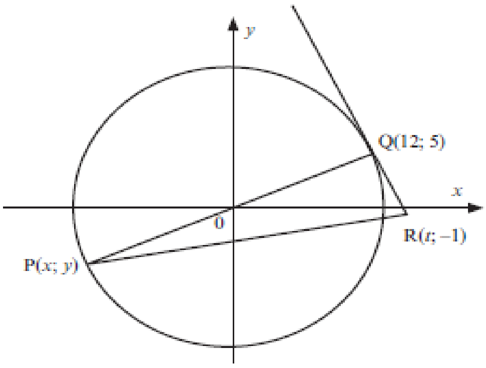
Draw a sketch to help you.
Centre of circle is C(3; –1) so the gradient of the radius CP (m_{CP})
is $\frac{3-(-1)}{5-3} = 2$.
radius $\perp$ tangent, so $m_{APB} \times m_{CP} = -1$ and so
$m_{APB} = -\frac{1}{2}$



Equation of tangent: $y - y_1 = m(x - x_1)$
$y - 3 = -\frac{1}{2}(x - 5)$ P is a point on the tangent
$y - 3 = -\frac{1}{2}x + 2\frac{1}{2}$
$y = -\frac{1}{2}x + 5\frac{1}{2}$

Example 7 (Try yourself)

2. O is the centre of the circle in the figure below. P(x; y) and Q(12; 5) are two points on the circle. POQ is a straight line. The point R(t; –1) lies on the tangent to the circle at Q.



2.1 Determine the equation of the circle. (3)

2.2 Determine the equation of the straight line through P and Q. (2)

2.3 Determine x and y, the coordinates of P. (4)

2.4 Show that the gradient of QR is $-\frac{12}{5}$. (2)

2.5 Determine the equation of the tangent QR in the form $y = \ldots$ (3)

2.6 Calculate the value of t. (2)

2.7 Determine an equation of the circle with centre Q(12; 5) and passing through the origin. (3)

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Answer

The centre is at the origin, so $x^2 + y^2 = r^2$.

2.1 $OQ^2 = (5)^2 + (12)^2 = 25 + 144 = 169$ ✓✓

So the equation of the circle is $x^2 + y^2 = 169$ ✓ (3)

2.2 $m_{PQ} = m_{OQ} = \frac{0-5}{0-12} = \frac{5}{12}$ ✓

PQ has y-intercept of 0. ✓ (2)

$y = \frac{5}{12}x$

2.3 By symmetry, P is the point (–12; –5). ✓✓ OR

Substitute $y = \frac{5}{12}x$ into $x^2 + y^2 = 169$

$x^2 + \left(\frac{5}{12}x\right)^2 = 169$

$x^2 + \frac{25}{144}x^2 = 169$

$144x^2 + 25x^2 = 169 \times 144$

$169x^2 = 24\,336$

$x^2 = 144$

$x = 12$ or $x = -12$ $\quad x = -12$ according to given diagram ✓

$y = \frac{5}{12}x = \frac{5}{12} \times (-12) = -5$ ✓ (4)

So P is the point (–12; –5).

2.4 tangent ⊥ radius so QR ⊥ PQ ✓

$m_{PQ} = \frac{0-5}{0-12} = \frac{5}{12}$

$\therefore m_{QR} = \frac{-12}{5}$ ✓ (2

2.5 $y = \frac{-12}{5}x + c$ ✓ OR $y - y_1 = \frac{-12}{5}(x - x_1)$ ✓

Substitute Q(12; 5) into equation to find c:

$5 = \frac{-12}{5}(12) + c$ ✓ $\qquad y - 5 = \frac{-12}{5}(x - 12)$ ✓

$5 + \frac{144}{5} = c$ $\qquad y = \frac{-12}{5}x + \frac{144}{5} + 5$

$c = \frac{169}{5}$ ✓ $\qquad y = -\frac{12}{5}x + \frac{169}{5}$ ✓

$y = \frac{-12}{5}x + \frac{169}{5}$ (3

2.6 R(t; –1) lies on line with equation $y = \frac{-12}{5}x + \frac{169}{5}$

$\therefore -1 = \frac{-12}{5}t + \frac{169}{5}$ ✓

$-5 = -12t + 169$

$12t = 174$

$t = 14{,}5$ ✓ (2

2.7 $OQ^2 = (x - 12)^2 + (y - 5)^2$ ✓✓ Q(12; 5) is centre of circle

Substitute (0; 0) into equation:

$OQ^2 = (0 - 12)^2 + (0 - 5)^2$

$OQ^2 = 144 + 25 = 169$ ✓

$\therefore (x - 12)^2 + (y - 5)^2 = 169$ (3

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