DIFFERENTIAL CALCULUS

12 1 7

FIRST PRINCIPLES

Visual representation of 1st Principles

Graphical Representation of Derivatives

- We are going to find the gradient at a point, anywhere along any function, using the First Principles Formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

First Principles Example:
f(x) = a (where a is a constant)

$f(x) = 3$

$\therefore f(x+h) = 3$ A CONSTANT so does NOT change

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3-3}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$= 0$

First Principles Example: f(x) = ax

$f(x) = 2x$

$f(x+h) = 2(x+h) = 2x + 2h$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2x + 2h - 2x}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h}$$

$= 2$

First Principles Example: $f(x) = ax^2$

$f(x) = x^2$

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - (x^2)}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$= 2x$

First Principles Example: $f(x) = ax^2$

$f(x) = 2x^2 - 1$

$f(x+h) = 2(x+h)^2 - 1 = 2x^2 + 4hx + 2h^2 - 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(2x^2 + 4hx + 2h^2 - 1) - (2x^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 4hx + 2h^2 - 1 - 2x^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{h(4x + 2h)}{h}$$

$= 4x$

Derivative of a Quadratic Function

First Principles Example: $f(x) = ax^3$

$f(x) = 2x^3$

$f(x+h) = 2(x+h)^3 = 2(x+h)(x^2+2xh+h^2)$

$= 2x^3 + 6x^2h + 6xh^2 + 2h^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(2x^3 + 6x^2h + 6xh^2 + 2h^3) - (2x^3)}{h}$$

$$= \lim_{h\to 0} \frac{(2x^3 + 6x^2h + 6xh^2 + 2h^3) - (2x^3)}{h}$$

$$= \lim_{h\to 0} \frac{h(6x^2 + 6xh + 2h^2)}{h} = 6x^2$$

Derivative of a Cubic Function

First Principles Example: $f(x) = \frac{a}{x}$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{x+h} - \frac{1}{x}\right) \div \frac{h}{1}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)} \times \frac{1}{h}$$

Fractions within fractions are a clumsy! So, use division signs

Get an LCD

First Principles Example continued .. $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{x - x - h}{x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{1}{x[x+(0)]}$$

$$= \frac{1}{x^2}$$

DIFFERENTIATION NOTATION

1. Question: $f(x) = \ldots$

   Differentiated Answer: $f'(x) = \ldots$

2. Question: $y = \ldots$

   Differentiated Answer: $\frac{dy}{dx} = \ldots$

3. Question: $D(x)\,[\ldots]$

   Answer: $= \ldots$

<u>Relationship between f(x) and f $^{\prime}$ (x)</u>

RULES OF DIFFERENTIATION

1. $f'(a) = 0$
2. $f'(ax) = a$
3. $f'(ax^n) = n.a\,x^{n-1}$

where a and n are constants

Differentiation Rules Examples

1. $f(x) = -2$

   $f'(x) = 0$

   $f'(a) = 0$

2. $f(x) = 4x$

   $f'(x) = 4$

   $f'(ax) = a$

3. $f(x) = -3x^2 + 6x - 9$

   $f'(x) = -6x + 6$

   $f'(ax^n) = n.ax^{n-1}$

Differentiation Rules Examples

4. $f(x) = (x-1)(x-2)$

$= x^2 - 3x + 2$

$f'(x) = 2x - 3$

First multiply out brackets

5. $f(x) = \frac{1}{x}$

$= x^{-1}$

$f'(x) = -x^{-2}$

Can't differentiate with x in the denominator … so write with negative exponents

6. $f(x) = \frac{x}{3} \; (= \frac{1}{3}x)$

$f'(x) = \frac{1}{3}$

Careful now! There is no x in the denominator

Differentiation terms in the form ax^n

Differentiation Rules Examples

7. $f(x) = \frac{8x^2 - x}{x}$

$= \frac{8x^2}{x} - \frac{x}{x}$

$= 8x - 1$

$f'(x) = 8$

Simplify by splitting terms!

8. $f(x) = \frac{x^2 - 9}{x - 3}$

$= \frac{(x-3)(x+3)}{x-3}$

$= x + 3$

$f'(x) = 1$

Simplify by factorization!

Differentiating terms with fractions

Differentiation Rules Examples

9. $f(x) = \sqrt{x}$

$= x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

Can't differentiate with roots, so write with rational exponents!

10. $f(x) = 4.\sqrt[3]{x}$

$= 4.x^{\frac{1}{3}}$

$f'(x) = \frac{4}{3}\ x^{-\frac{2}{3}}$

Careful now with the 4 !

Differentiating terms with roots

Recap of the concept of the derivative