



**DIFFERENTIAL
CALCULUS**

FIRST PRINCIPLES

Visual representation of 1st Principles

Graphical Representation of Derivatives

➤ We are going to find the gradient at a point, anywhere along any function, using the First Principles Formula:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

First Principles Example: $f(x) = a$ (where a is a constant)

$$f(x) = 3$$

$\therefore f(x+h) = 3$ A CONSTANT so does NOT change

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3 - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0}{h}$$

$$= 0$$

First Principles Example:

$$f(x) = ax$$

$$f(x) = 2x$$

$$f(x+h) = 2(x+h) = 2x + 2h$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h}{h}$$

$$= 2$$

First Principles Example:

$$f(x) = ax^2$$

$$f(x) = x^2$$

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 - (\cancel{x^2})}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} \\ &= 2x \end{aligned}$$

First Principles Example:

$$f(x) = ax^2$$

$$f(x) = 2x^2 - 1$$

$$f(x+h) = 2(x+h)^2 - 1 = 2x^2 + 4hx + 2h^2 - 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2x^2 + 4hx + 2h^2 - 1) - (2x^2 - 1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4hx + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(4x + 2h)}{h}$$

$$= 4x$$

Derivative of a Quadratic
Function

First Principles Example:

$$f(x) = ax^3$$

$$f(x) = 2x^3$$

$$\begin{aligned} f(x+h) &= 2(x+h)^3 = 2(x+h)(x^2+2xh+h^2) \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2x^3 + 6x^2h + 6xh^2 + 2h^3) - (2x^3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{(2x^3)} + 6x^2h + 6xh^2 + 2h^3 - \cancel{(2x^3)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x^2 + 6xh + 2h^2)}{h} = 6x^2 \end{aligned}$$

Derivative
of a Cubic
Function

First Principles Example:

$$f(x) = \frac{a}{x}$$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{1}{x+h} - \frac{1}{x} \right) \div \frac{h}{1}$$

$$= \lim_{h \rightarrow 0} \frac{x - (x+h)}{x(x+h)} \times \frac{1}{h}$$

Fractions within fractions are a clumsy! So, use division signs

Get an LCD

First Principles Example continued .. $f(x) = \frac{1}{x}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{x - (x + h)}{x(x + h)} \times \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{x - x - h}{x(x + h)} \times \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{x(x + h)} \times \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x + h)} \\ &= \frac{1}{x[x + (0)]} \\ &= \frac{1}{x^2} \end{aligned}$$

DIFFERENTIATION NOTATION

1. Question: $f(x) = \dots$
Differentiated Answer: $f'(x) = \dots$
2. Question: $y = \dots$
Differentiated Answer: $\frac{dy}{dx} = \dots$
3. Question: $D(x) [\dots]$
Answer: $= \dots$

Relationship between $f(x)$ and $f'(x)$

RULES OF DIFFERENTIATION

$$1. \quad f'(a) = 0$$

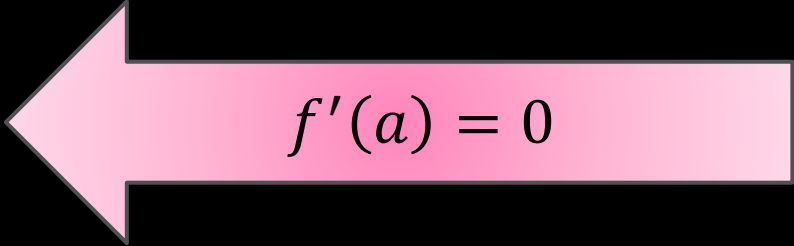
$$2. \quad f'(ax) = a$$

$$3. \quad f'(ax^n) = n \cdot a x^{n-1}$$

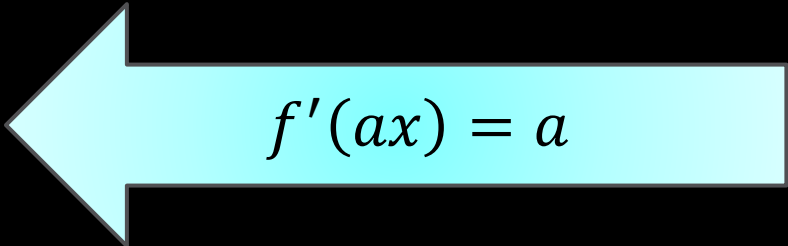
where a and n are constants

Differentiation Rules Examples

1. $f(x) = -2$
 $f'(x) = 0$


$$f'(a) = 0$$

2. $f(x) = 4x$
 $f'(x) = 4$


$$f'(ax) = a$$

3. $f(x) = -3x^2 + 6x - 9$
 $f'(x) = -6x + 6$


$$f'(ax^n) = n \cdot ax^{n-1}$$

Differentiation Rules Examples

4. $f(x) = (x - 1)(x - 2)$

$$= x^2 - 3x + 2$$

$$f'(x) = 2x - 3$$

First multiply out brackets

5. $f(x) = \frac{1}{x}$

$$= x^{-1}$$

$$f'(x) = -x^{-2}$$

Can't differentiate with x in the denominator ... so write with negative exponents

6. $f(x) = \frac{x}{3} (= \frac{1}{3}x)$

$$f'(x) = \frac{1}{3}$$

Careful now! There is no x in the denominator

Differentiation terms in the form ax^n

Differentiation Rules Examples

$$\begin{aligned} 7. \quad f(x) &= \frac{8x^2 - x}{x} \\ &= \frac{8x^2}{x} - \frac{x}{x} \\ &= 8x - 1 \end{aligned}$$

$$f'(x) = 8$$

Simplify by splitting terms!

$$\begin{aligned} 8. \quad f(x) &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x - 3)(x + 3)}{x - 3} \\ &= x + 3 \end{aligned}$$

$$f'(x) = 1$$

Simplify by factorization!

Differentiating terms with fractions

Differentiation Rules Examples

$$9. \quad f(x) = \sqrt{x} \\ = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

Can't differentiate with roots, so write with rational exponents!

$$10. \quad f(x) = 4 \cdot \sqrt[3]{x} \\ = 4 \cdot x^{\frac{1}{3}}$$

$$f'(x) = \frac{4}{3} x^{-\frac{2}{3}}$$

Careful now with the 4!

Differentiating terms with roots

Recap of the concept of the derivative